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IETI-DP for Conforming Multi-Patch Isogeometric Analysis in Three Dimensions

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IETI-DP for conforming multi-patch Isogeometric Analysis in three dimensions

Rainer Schneckenleitner and Stefan Takacs*

Abstract We consider dual-primal isogeometric tearing and interconnection (IETI-DP) solvers for multi-patch geometries in Isogeometric Analysis. Recently, the authors have published a convergence analysis for those solvers that is explicit in both the grid size and the spline degree for conforming discretizations of two dimensional computational domains. In the present paper, we shortly revisit these results and provide numerical experiments that indicate that similar results may hold for three dimensional domains.

1 Introduction

We are interested in fast domain decomposition solvers for multi-patch Isogoemtric Analysis (IgA; [4]). We focus on variants of FETI-DP solvers, see [2, 10] and references therein. Such methods have been adapted to IgA in [5], where the individual patches of the multi-patch discretization are used as subdomains for the solver. This method is sometimes referred to as the dual-primal isogeometric tearing and interconnection (IETI-DP) method. These methods are similar to Balancing Domain Decomposition by Constraints (BDDC) methods, which have also been adapted for IgA, see [1] and follow-up papers by the same authors. The similarity is outlined in [6].

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Much progress for the IETI-DP methods has been made in the PhD-thesis by C. Hofer, including the extension to various discontinuous Galerkin formulations, see [3]. Recently, the authors of the paper at hand have extended the condition number bounds for the preconditioned Schur complement system to be explicit not only in the grid size but also in the chosen spline degree, see [8] for the conforming case and [9] for an extension to the discontinuous Galerkin case. The analysis follows the framework from [6]. One key ingredient for the analysis in [8] has been the construction of a bounded harmonic extension operator for splines, which followed the ideas of [7]. The analysis in [8] treats the two-dimensional case. As usual for FETI-like methods, the extension of the analysis to three dimensions is not effortless. The goal of this paper is to demonstrate that the proposed method also performs well for higher spline degrees in three dimensions.

The remainder of this paper is organized as follows. In Section 2, we introduce the model problem, discuss its discretization and the proposed IETI-DP algorithm. In Section 3, numerical experiments for a three dimensional example are presented.

2 Model problem and its solution

We consider a standard Poisson model problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For given $f \in L_2(\Omega)$, we are interested in solving for $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega$$

holds in a weak sense. We assume that the closure of the computational domain Ω is the union of the closure of k non-overlapping patches $\Omega^{(k)}$ that are parametrized with geometry functions

$$G_k : \widehat{\Omega} := (0,1)^d \to \Omega^{(k)} := G_k(\widehat{\Omega})$$

such that for any $k \neq \ell$, the intersection $\overline{\Omega^{(k)}} \cap \overline{\Omega^{(\ell)}}$ is empty, a common vertex, a common edge, or (in three dimensions) a common face (cf. [8, Ass. 2]). We assume that both, ∇G_k and $(\nabla G_k)^{-1}$, are in $L_\infty(\widehat{\Omega})$ for all patches. For the analysis, we need a uniform bound on the L_∞-norm and a uniform bound on the number of neighbors of each patch, cf. [8, Ass. 1 and 3].

For each of the patches, we introduce a tensor B-spline discretization on the parameter domain $\widehat{\Omega}$. The discretization is then mapped to the physical patch $\Omega^{(k)}$ using the pull-back principle. We use a standard basis as obtained by the Cox-de Boor formula. We need a fully matching discretization, this means that for each basis function that has a non-vanishing trace on one

of the interfaces, there is exactly one basis function on each of the patches sharing this interface such that the traces of the basis functions agree (cf. [8, Ass. 5]). This is a standard assumption for any multi-patch setting that is not treated using discontinuous Galerkin methods. For the analysis, we assume quasi uniformity of grids within each patch, cf. [8, Ass. 4].

In the following, we explain how to the IETI-DP solver is set up. Here, we loosely follow the notation used in the IETI-DP solution framework that recently joined the public part of the G+Smo library. We choose the patches as IETI subdomains. We obtain patch-local stiffness matrices $A^{(k)}$ by evaluating the bilinear forms $a^{(k)}(u,v) = \int_{\Omega^{(k)}} \nabla^\top u(x) \nabla v(x) \mathrm{d}x$ with the basis functions for the corresponding patch. We set up matrices $C^{(k)}$ such that their null space are the coefficient vectors of the patch-local functions that vanish at the primal degrees of freedom. In [8], we have considered corner values, edge averages, and the combination of both. In the three dimensional case, we can choose corner values, edge averages, face averages, and any combination thereof. We set up fully redundant jump matrices $B^{(k)}$. We omit the corner values if and only if the corners are chosen as primal degrees of freedom. We setup the primal problem in the usual way, i.e., we first, for $k = 1, \dots, K$, compute a basis by

$$\Psi^{(k)} := \begin{pmatrix} I & 0 \end{pmatrix} (\widetilde{A}^{(k)})^{-1} \begin{pmatrix} 0 \\ R_c^{(k)} \end{pmatrix}, \quad \text{where} \quad \widetilde{A}^{(k)} := \begin{pmatrix} A^{(k)} & (C^{(k)})^\top \\ C^{(k)} & \end{pmatrix}$$

and $R_c^{(k)}$ is a binary matrix that relates the primal constraints (with their patch-local indices) to the degrees of freedom of the primal problem (with their global indices) and set then

$$\widetilde{A}^{(K+1)} := \sum_{k=1}^{K} (\Psi^{(k)})^\top A^{(k)} \Psi^{(k)}, \quad \text{and} \quad \widetilde{B}^{(K+1)} := \sum_{k=1}^{K} B^{(k)} \Psi^{(k)}.$$

We consider the Schur complement problem $F \underline{\lambda} = \underline{g}$, where

$$F := \sum_{k=1}^{K+1} \widetilde{B}^{(k)} (\widetilde{A}^{(k)})^{-1} (\widetilde{B}^{(k)})^\top \quad \text{and} \quad \widetilde{B}^{(k)} := \begin{pmatrix} B^{(k)} & 0 \end{pmatrix} \text{ for } k = 1, \dots, K.$$

The derivation of $\underline{g}$ is a patch-local preprocessing step. We solve the Schur complement problem using a preconditioned conjugate gradient (PCG) solver with the scaled Dirichlet preconditioner

$$M_{\mathrm{sD}} := \sum_{k=1}^{K} B_\Gamma D_k^{-1} \Big(A_{\Gamma\Gamma}^{(k)} - A_{\Gamma I}^{(k)} (A_{II}^{(k)})^{-1} A_{I\Gamma}^{(k)} \Big) D_k^{-1} (B_\Gamma)^\top,$$

where the index Γ refers to the rows/columns of $A^{(k)}$ and the columns of $B^{(k)}$ that refer to basis functions with non-vanishing trace, the index I refers to the remaining rows/columns, and the matrix D_k is a diagonal matrix defined based on the principle of multiplicity scaling. For the analysis, it is important that its coefficients are constant within each interface. The solution u itself is obtained from $\underline{\lambda}$ using the usual patch-local steps, cf. [8].

Under the presented assumptions, the condition number of the preconditioned Schur complement system is in the two-dimensional case bounded by

$$C\,p\,\left(1+\log p+\max_{k=1,\dots,K}\log\frac{H_k}{h_k}\right)^2,$$

where p is the spline degree, H_k is the patch size, and h_k the grid size, see [8].

3 Numerical results

In the following, we present numerical results for a three dimensional domain and refer to the original paper [8] for the two dimensional case. The computational domain Ω is a twisted version of a Fichera corner, see Fig. 1. The original geometry consists of 7 patches. We subdivide each patch uniformly into $4 \times 4 \times 4$ patches to obtain a decomposition into 448 patches.

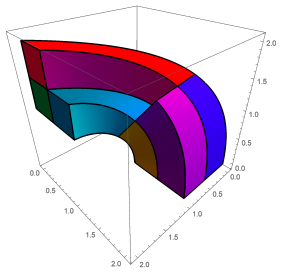

Fig. 1: Computational domain

We solve the model problem $-\Delta u(x,y,z) = 3\pi^2 \sin(\pi x)\sin(\pi y)\sin(\pi z)$ for $(x,y,z) \in \Omega$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$ by means of the IETI-DP solver outlined in the previous sections. Within the patches, we consider tensor-product B-spline discretizations of degree p and maximum smoothness C^{p-1}. We consider several grid sizes, the refinement level $r = 0$ corresponds to a discretization of each patch with polynomials. The next refinement levels $r = 1, 2, \dots$ are obtained by uniform refinement.

All experiments have been carried out in the C++ library G+Smo[1] and have been executed on the Radon1 cluster[2] in Linz. All computations have been performed with a single core.

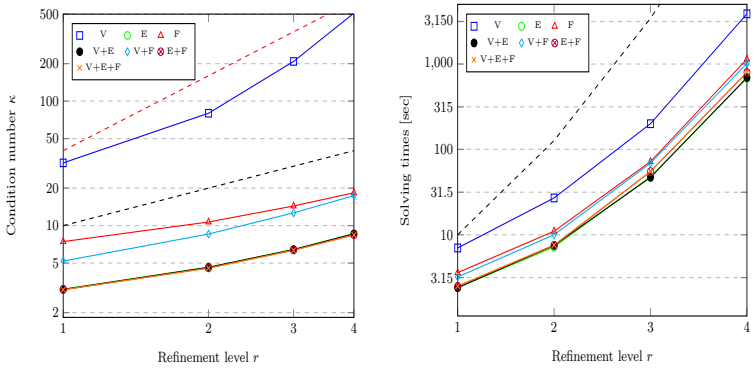

Fig. 2: Condition numbers and solving times for $p = 3$

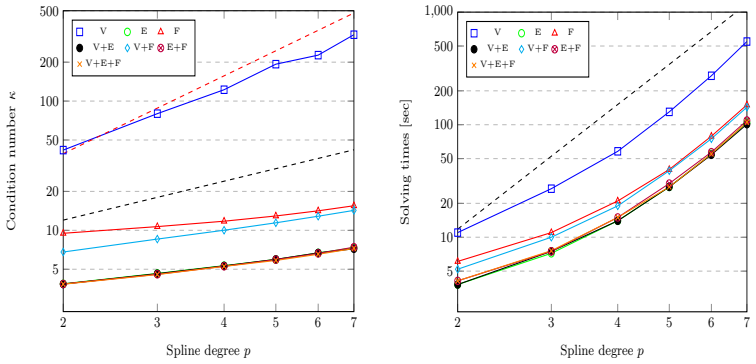

Fig. 3: Condition numbers and solving times for $r = 2$

Concerning the choice of the primal degrees of freedom, we consider all possibilities. For the two-dimensional case, the common choices are the corner values, the edge averages, and a combination of both. We have seen in [8] that all approaches work, typically the corner values better than the edge averages. As expected, the combination of both yields the best results. For the three dimensional case, we have more possibilities. We report on these approaches

[1] `https://github.com/gismo/gismo`, example file `examples/ieti_example.cpp`.

[2] `https://www.ricam.oeaw.ac.at/hpc/`

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	33	14	51	32	64	45	89	84	108	109	136	178
2	57	42	79	80	98	122	124	193	148	227	176	326
3	94	116	123	208	149	315	175	439	199	566	OoM	
4	146	275	176	509	OoM		OoM		OoM		OoM	

Table 1: Iterations (it) and condition number (κ); Vertex (V)

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	14	2.5	17	3.1	20	3.8	23	4.4	27	5.1	29	5.5
2	18	3.9	21	4.6	23	5.3	26	6.0	29	6.7	32	7.3
3	23	5.6	25	6.4	28	7.3	30	8.0	33	8.8	OoM	
4	27	7.5	30	8.6	OoM		OoM		OoM		OoM	

Table 2: Iterations (it) and condition number (κ); Edges (E)

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	22	6.1	26	7.4	29	8.3	33	9.5	37	10.4	41	11.5
2	29	9.5	31	10.7	34	11.8	37	12.9	42	14.2	46	15.5
3	35	13.1	38	14.4	41	15.9	43	17.0	47	18.3	OoM	
4	41	17.1	44	18.4	OoM		OoM		OoM		OoM	

Table 3: Iterations (it) and condition number (κ); Faces (F)

in the Tables 1 (vertex values = V), 2 (edge averages = E), 3 (face averages = F), 4 (V+E), 5 (V+F), and 6 (E+F). The combination of all variants (V+E+F) is almost identical to the case V+E and only included in the diagrams. In any case, we report on the number of iterations (it) required by the PCG solver to reduce the residual with a random starting vector by a factor of 10^{-6} compared to the right-hand side. Moreover, we report on the condition numbers (κ) of the preconditioned system as estimated by the PCG solver.

In Figure 2, the dependence on the refinement level is depicted. Here, we have chosen the spline degree $p = 3$ and have considered all of the possibilities for primal degrees of freedom. Here, we have 44 965 ($r = 1$), 133 629 ($r = 2$), 549 037 ($r = 3$), and 2 934 285 ($r = 4$) degrees of freedom (dofs). We observe that choosing only vertex values as primal degrees of freedom leads to the largest condition numbers. We observe that in this case the con-

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	14	2.5	17	3.1	20	3.8	22	4.3	26	5.0	28	5.4
2	18	3.8	21	4.6	23	5.3	26	6.0	29	6.7	31	7.2
3	22	5.5	25	6.4	28	7.3	30	8.0	33	8.8	OoM	
4	27	7.5	30	8.6	OoM		OoM		OoM		OoM	

Table 4: Iterations (it) and condition number (κ); Vertices+Edges (V+E)

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	17	3.7	22	5.2	26	6.6	30	7.8	34	9.0	38	10.1
2	25	6.8	29	8.6	32	10.0	36	11.4	40	12.9	44	14.3
3	32	10.7	36	12.7	39	14.2	42	15.6	45	17.3	OoM	
4	39	15.2	43	17.4	OoM		OoM		OoM		OoM	

Table 5: Iterations (it) and condition number (κ); Vertices+Face (V+F)

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	14	2.5	17	3.1	20	3.8	23	4.3	27	5.0	30	5.7
2	19	3.9	21	4.6	24	5.3	27	5.9	30	6.6	33	7.4
3	23	5.5	26	6.4	29	7.2	31	7.9	33	8.5	OoM	
4	28	7.4	31	8.4	OoM		OoM		OoM		OoM	

Table 6: Iterations (it) and condition number (κ); Edges+Faces (E+F)

dition number grows like r^2 (the dashed red line indicates the slope of such a growth). This corresponds to a growth like $(1+\log H/h)^2$, as predicted by the theory for the two-dimensional case. All other options yield significantly better results, particularly those that include edge averages. In these cases, the growth seems to be less than linear in $r \approx \log H/h$ (the dashed black like shows such a slope). In the right diagram, we can see that choosing a strategy with smaller condition numbers also yields a faster method. Since the dimensions and the bandwidths of the local stiffness matrices grow like $(H_k/h_k)^3$ and $(H_k/h_k)^2$, respectively, the complexity of the LU decompositions grows like $\sum_{k=1}^{K}(H_k/h_k)^7$. The complexity analysis indicates that they are the dominant factor. The dashed black line indicates such a growth.

In Figure 3, the dependence on the spline degree is presented, where we have chosen $r=2$. Here, the number of dofs ranges from 66 989 ($p=2$) to

549 037 ($p = 7$). Also in this picture, we see that the vertex values perform worst and the edge averages best. Again, we obtain a different asymptotic behavior for the corner values. For those primal degrees of freedom, the condition number grows like p^2 (the dashed red line indicates the corresponding slope). All the other primal degrees of freedom seem to lead to a growth that is smaller than linear in p (the dashed black line indicates the slope of a linear growth). Note that for the two-dimensional case, the theory predicts a growth like $p(1 + \log p)^2$. In the right diagram, we can see that the solving times grow like p^4 (the dashed line shows the corresponding slope). This seems to be realistic since the number of non-zero entries of the stiffness matrix grows like Np^d, where N is the number of unknowns. For $d = 3$, this yields in combination with the condition number bound the observed rates.

Concluding, in this paper we have seen that the IETI method as described in [8] can indeed be extended to the three dimensional case. As for finite elements, only choosing vertex values is not enough.

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