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Convergence Estimates of Finite Elements for a Class of Quasilinear Elliptic Problems

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Convergence estimates of finite elements for a class of quasilinear elliptic problems

S. Nakov*, I. Toulopoulos†

Abstract

This paper is concerned with conforming finite element discretizations for quasilinear elliptic problems in divergence form, of a class that generalizes the p-Laplace equation and allows to show existence and uniqueness of the continuous and discrete problems. We derive discretization error estimates under general regularity assumptions for the solution and using high order polynomial spaces, resulting in convergence rates that are then verified numerically. A key idea of this error analysis is to consider carefully the relation between the natural $W^{1,p}$-seminorm and a specific quasinorm introduced in "L. Diening and M. Růžička. *Numer. Math.*, 107(1):107–129, 2007". In particular, we are able to derive interpolation estimates in this quasinorm from known interpolation estimates in the $W^{1,p}$-seminorm. We also give a simplified proof of known near-best approximation results in $W^{1,p}$-seminorm starting from the corresponding result in the mentioned quasinorm.

Keywords: quasilinear elliptic equations, p-Laplace, power-law diffusion, finite element, near-best approximation results, discretization error analysis, a priori error estimates, quasi-norm estimates.

MSC 2020: 65N30, 65J15, 65N15

1 Introduction

The study of quasilinear elliptic equations of the form [9],

$$-\mathrm{div}\big(\mathbf{A}(\nabla u)\big) = f \quad \text{in } \Omega \subset \mathbb{R}^d,\ d = 2, 3, \tag{1.1a}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\mathbf{A}$, g and f are given functions (see their properties below), is of great importance since these equations are used to describe many physical problems arising in the area of non-Newtonian motions, porous media, chemical engineering etc, [28, 30, 33]. Basic examples of (1.1) are variations of the p-Laplace model where $\mathbf{A}(\nabla u) = (\epsilon + |\nabla u|)^{p-2}\nabla u$ with $\epsilon \geq 0$ and $p > 1$. The first analysis of finite element

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(FE) methods for p-Laplace ($\epsilon = 0$) was undertaken in [29] and in [11] (Chapter 5), where (sub-optimal) a priori error estimates have been shown in the $W^{1,p}$-norm. These results were later improved in [32, 4].

In the literature there are various works discussing sufficient conditions on the regularity of the boundary data g and the righ-hand side f which guarantee certain regularity properties of the solutions of p-type problems. Here, by p-type problems we refer to problems, in which the nonlinear operator $\mathbf{A}$ has some "p-structure" properties that guarantee solvability in the classical Sobolev space $W^{1,p}(\Omega)$ for some $p > 1$. The nonlinear nature of these quasilinear problems usually results in the existence of weak solutions with reduced regularity even for very smooth problem data (see, e.g., [9, 37]). In general, the regularity of the solutions that can be inferred is only enough to show optimal error estimates using linear finite element spaces, see, e.g., [36, 34, 5]. Perhaps this is the main reason for the predominant use of linear FE spaces for the discretization of such problems. This is in contrast to linear problems, where the higher the regularity of the input data is, the higher the regularity of the corresponding solution is. However, applications of k and hk finite element methods for p-Laplace problems have been recently presented, where h denotes the spatial discretization parameter and k the degree of the polynomial space, [23, 24]. We mention that the presence of the gradient in the non-linear diffusion term leads to algebraic systems with bad condition number, [14], which requires efficient iterative techniques, [27, 20].
Over the last two decades, there has been an increasing interest on devising discontinuous Galerkin (DG) methods in addition to the more popular continuous ones for the numerical solution of (1.1) with basic examples to be certain p-type problems, see [18, 8, 13]. In all these DG methods the numerical fluxes were developed by following the p-nonlinear nature of the problem. The quasi-norm interpolation estimates presented in [21], were applied in the framework of broken spaces and optimal a priori error estimates were shown for linear elements. The same DG methods have been used later in [7] for solving non-Newtonian flow problems.

In this paper, we focus on problems where the function $\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^d$ in (1.1), has the form

$$\mathbf{A}(\mathbf{a}) = \varphi'(|\mathbf{a}|)\frac{\mathbf{a}}{|\mathbf{a}|} \text{ for all } \mathbf{a} \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \qquad \mathbf{A}(\mathbf{0}) = \mathbf{0}, \tag{1.2}$$

where φ' is the right derivative of an N-function φ (see Definition 2.1). We start our investigation by introducing certain assumptions on the function φ, which guarantee existence and uniqueness of the solution of the continuous problem in the framework of Sobolev $W^{1,p}$-spaces by using classical results of the calculus of variations, e.g., convexity, lower semicontinuity and coercivity of the associated energy functional. We continue with our main objective in this work, which is to present continuous FE approximations of (1.1) and to derive a priori estimates for the error $u - u_h$ where u_h is the corresponding finite element solution. Motivated by the results in [21], we present an analysis of higher order finite element approximations for the problem (1.1) where $\mathbf{A}$ has the structure given in (1.2). It has been explained in [21, 26] that for the study of (1.1), it is appropriate to introduce the closely related to the operator $\mathbf{A}$ function $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$

$$\mathbf{F}(\mathbf{a}) = (\varphi'(|\mathbf{a}|)\,|\mathbf{a}|)^{\frac{1}{2}}\,\frac{\mathbf{a}}{|\mathbf{a}|} \text{ for all } \mathbf{a} \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \qquad \mathbf{F}(\mathbf{0}) = \mathbf{0}. \tag{1.3}$$

We derive a priori discretization error estimates for the error $u - u_h$ measured by the $\mathbf{F}$-quantity $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$ which is closely related to the so-called *quasinorm*

$$\|\nabla u - \nabla u_h\|_{(u,p)} := \left(\int_\Omega (\epsilon + |\nabla u| + |\nabla(u - u_h)|)^{p-2} |\nabla(u - u_h)|^2 \, dx \right)^{\frac{1}{2}}$$

introduced by Barrett and Liu in [35] when the nonlinear operator $\mathbf{A}$ has a p-structure (see Assumption 2 below). More precisely, for operators with p-structure, the above two quantities are equivalent (see, e.g., [6, 21]) and thus error estimates and near-best approximation results derived in terms of one of them immediately translate to error estimates in terms of the other. Therefore, the a priori error estimates and convergence rates derived in this paper are also valid for the discretization error measured by the quasinorm of Barrett and Liu. For short, we will loosely refer to the measure generated by $\mathbf{F}$, i.e. $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$, as $\mathbf{F}$-quasinorm.

Even though showing solvability in Orlicz-Sobolev spaces is a more general approach (see, e.g., [6]), we choose to work with standard Sobolev spaces (by assuming a p-structure of the operator $\mathbf{A}$, see Assumption 2 below) so that we can relate the $\mathbf{F}$-quasinorm with the Sobolev $W^{1,p}$-seminorm and use known interpolation estimates in this seminorm (see, e.g., [31]). In particular, we utilize the near-best approximation result from [21] with respect to the $\mathbf{F}$-quasinorm and bound the approximation error of the FE space in terms of approximation errors $|u - v_h|_{W^{1,q}(\Omega)}$ for some $1 \leq q \leq \infty$ (see (4.5), (4.7), (4.10)). By taking v_h to be the finite element interpolant $I_h u$ in these bounds, we arrive at the respective interpolation error estimates in the $\mathbf{F}$-quasinorm under regularity assumptions of the form $u \in W^{l,q}(\Omega)$ for some $l \geq 2$ and $q \geq 1$ (see (4.6), (4.8), (4.11), (4.12) and Theorem 4.2). Thus, from the interpolation estimates just mentioned, we also obtain the regularity conditions that guarantee optimal (with respect to the approximation error and polynomial degree) discretization convergence rates.

In the current work we do not investigate conditions on the data that ensure high regularity solutions. An analysis in this direction for homogeneous right-hand side is presented for example in [36]. As we mentioned earlier, it is in general not possible to guarantee higher regularity of the solutions even for very smooth data and for this reason the h-version of the finite element method with a fixed first order polynomial degree is the most commonly used one. However, deriving approximation error estimates in the $\mathbf{F}$-quasinorm for high order polynomial spaces can be useful, for example, in the development of adaptive FEM (see, e.g., [17] for adaptive FEM with polynomial degree one and error measured in the $\mathbf{F}$-quasinorm). Those estimates give a benchmark for the approximation properties of the FE space with respect to the quasinorm in question and display the convergence order at which the adaptive methods should be aming at.

Here, we emphasize that our regularity assumptions are posed directly on the solution u itself, rather than on the function $\mathbf{F}(\nabla u)$. This differs from the approach presented in [21], where the authors prove a priori error estimates in a more general setting, but optimal only for first order polynomial FE spaces and under the regularity assumption $\mathbf{F}(\nabla u) \in [W^{1,2}(\Omega)]^d$. To the best of our knowledge, approximation estimates of this type, including higher order finite element discretizations, have not yet been presented in the existing literature.

The layout of the paper is as follows. In the next section we study the existence and uniqueness of the solution of the partial differential equation. In Section 3 we present the finite element discretization of the problem and we give a simple proof of known near best approximation results in the $W^{1,p}$ seminorm by carefully examining the relation between the **F**-quasinorm and the associated Sobolev $W^{1,p}$-seminorm, and by making use of the near-best approximation result in terms of the **F**-quasinorm from [21]. In Section 4 we give interpolation error estimates with respect to the **F**-quasinorm under various regularity assumptions on the solution u. We also discuss the sufficient regularity conditions that guarantee optimal convergence rates with respect to the polynomial degree k of the FE space. The paper closes with Section 5, where an extensive series of numerical tests is given which confirm the theoretically predicted convergence rates.

2 Preliminaries and problem formulation

2.1 Notations and known inequalities

We use a standard notation throughout this work. Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ be a bounded Lipschitz domain. For $1 \leq p < \infty$ we denote by $L^p(\Omega)$ the space of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|^p_{L^p(\Omega)} := \int_\Omega |u(x)|^p dx < \infty.$$

When $p = \infty$, then $L^\infty(\Omega)$ denotes the space of all measurable essentially bounded functions and is endowed with the norm

$$\|u\|_{L^\infty(\Omega)} := \inf\{C > 0 \text{ s.t. } |u(x)| \leq C \text{ for almost each } x \in \Omega\}.$$

If $\mathbf{f}$ is a vector function in the Lebesgue space $[L^p(\Omega)]^d$, we use the norm

$$\|\mathbf{f}\|_{L^p(\Omega)} := \|\,|\mathbf{f}|\,\|_{L^p(\Omega)} = \left(\int_\Omega |\mathbf{f}|^p\,dx\right)^{\frac{1}{p}}, \tag{2.1}$$

where $|\mathbf{a}|$ denotes the Eucledian ℓ_2 norm of $\mathbf{a} \in \mathbb{R}^d$.

Let l be a non-negative integer, and let $\alpha = (\alpha_1, \ldots, \alpha_d)$ with order $|\alpha| = \sum_{j=1}^d \alpha_j$. We define the differential operator $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x^{\alpha_d}}$ and denote the standard Sobolev spaces by $W^{l,p}(\Omega)$, which consist of the functions $u : \Omega \to \mathbb{R}$ such that their weak derivatives $D^\alpha u$ with $|\alpha| \leq l$ belong to $L^p(\Omega)$. The space $W^{l,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{l,p}(\Omega)} = \Big(\sum_{0\leq|\alpha|\leq l} \|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}} \quad \text{for} \quad 1 \leq p < \infty, \tag{2.2}$$

$$\|u\|_{W^{l,\infty}(\Omega)} = \max_{0\leq|\alpha|\leq l} \|D^\alpha u\|_{L^\infty(\Omega)} \quad \text{for } p = \infty. \tag{2.3}$$

For convenience, in the case $l = 1$, instead of the norm (2.2), we will be using the equivalent norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}},$$

where the term $\|\nabla u\|^p_{L^p(\Omega)}$ is computed with the norm (2.1) for the space $[L^p(\Omega)]^d$. For a function $u \in W^{1,p}(\Omega)$ will define its seminorm by

$$|u|_{W^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}, \tag{2.4}$$

where, again, the quantity on the right-hand side of (2.4) is computed by using the norm in $[L^p(\Omega)]^d$ given by (2.1). Furthermore, for $1 < p < \infty$ we denote by γ_p the trace operator $\gamma_p : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$,with the properties (i) $\gamma_p(u) = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, and (ii) $\|\gamma_p(u)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$. For a function $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we define the set

$$W^{1,p}_g := \{u \in W^{1,p}(\Omega) : \gamma_p(u) = g\}. \tag{2.5}$$

We refer the reader to [1, 22] for more details on Sobolev spaces. The following inequalities are going to be used in several places in the text. For functions $u \in W^{1,p}_0(\Omega)$ we have Poincaré's inequality

$$\|u\|_{L^p(\Omega)} \leq C_P\|\nabla u\|_{L^p(\Omega)} \tag{2.6a}$$

with a constant $C_P > 0$, depending on p and Ω. Next, let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, there holds Hölder's inequality:

$$\left|\int_\Omega uv\,dx\right| \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)} \tag{2.6b}$$

Reverse Hölder's inequality (see, e.g., Theorem 2.12 in [1]) reads: Let $0 < p < 1$ and $q = \frac{p}{p-1} < 0$. If $f \in L^p(\Omega)$ and $0 < \int_\Omega |g(x)|^q\,dx < \infty$ then

$$\int_\Omega |f(x)g(x)|\,dx \geq \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{2.6c}$$

Young's inequality: Let $0 < p < \infty$ and $q = \frac{p}{p-1}$. Then

$$ab \leq \frac{\delta a^p}{p} + \frac{b^q}{q\delta^{\frac{q}{p}}} \quad \text{for all } a, b \geq 0. \tag{2.6d}$$

Finally, for $a, b \in \mathbb{R}$ and $s > 0$ we have

$$|a + b|^s \leq C(s)\left(|a|^s + |b|^s\right), \tag{2.6e}$$

where $C(s) = 1$ if $0 < s < 1$ and $C(s) = 2^{s-1}$ if $s > 1$. Obviously, one can take $C(s) = 2^s$ regardless of whether $0 < s < 1$ or $s > 1$.

In what follows, for $a, b \geq 0$ we will frequently write $a \sim b$ meaning that $ca \leq b \leq Ca$ for some $c, C > 0$.

2.2 Properties of the given functions

In (1.1) we assume that $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ is a bounded domain with Lipschitz boundary $\partial\Omega$, the functions $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ and $f \in L^{p'}(\Omega)$ with $1/p + 1/p' = 1$, and the operator $\mathbf{A}$ is defined by (1.2), where φ is an N-function.

Definition 2.1 (see, e.g. [2, 1]). *A function $\varphi : [0,\infty) \to [0,\infty)$ is called an N-function if it is convex, continuous and satisfies $\lim\limits_{t\to 0} \frac{\varphi(t)}{t} = 0$, $\lim\limits_{t\to\infty} \frac{\varphi(t)}{t} = \infty$, and $\varphi(t) > 0$ for $t > 0$.*

Note that every N-function has a right derivative φ', which is right continuous, non-decreasing, and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$ and $\lim\limits_{t\to\infty} \varphi'(t) = \infty$ (see, e.g., [2, 1]). Moreover, the following integral representation of φ holds

$$\varphi(t) = \int_0^t \varphi'(s)ds. \tag{2.7}$$

Equivalently, one can define an N-function φ through the formula (2.7) with a function φ' having the above properties (see [2, 1]). In our considerations we will be only interested in N-functions $\varphi \in C^1[0,\infty) \cap C^2(0,\infty)$ with strictly increasing φ'. In this case, the complementary (Fenchel conjugate) function φ^* of φ is defined by

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)ds, \tag{2.8}$$

where $(\varphi')^{-1}$ denotes the inverse function of φ'. It is clear that φ^* is also an N-function. Since an N-function φ is convex with $\varphi(0) = 0$, it is superadditive and thus it holds $2\varphi(t) \leq \varphi(2t)$. We will be interested in functions φ that also satisfy the inequality $\varphi(2t) \leq K\varphi(t)$ with some constant $K \geq 2$. Such functions φ are said to satisfy the Δ_2-condition. In this case the Δ_2-constant of φ is the smallest constant K with the above property and is denoted by $\Delta_2(\varphi)$. For a family of N-functions $\{\varphi_\lambda\}_\lambda$ we define $\Delta_2(\{\varphi_\lambda\}_\lambda) := \sup_\lambda \Delta_2(\varphi_\lambda)$. Notice that for every N-function φ we have

$$\varphi(t) = \int_0^t \varphi'(s)ds \leq t\varphi'(t) \text{ for any } t \geq 0.$$

On the other hand, if φ also satisfies the Δ_2-condition, we obtain

$$\Delta_2(\varphi)\varphi(t) \geq \varphi(2t) \geq \int_t^{2t} \varphi'(s)ds \geq t\varphi'(t) \text{ for any } t \geq 0,$$

and thus, we see that $\varphi(t) \sim t\varphi'(t)$ uniformly in $t \geq 0$.

Assumption 1 (Assumption 5.1 in [21]). Let φ be an N-function with strictly increasing φ' and $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Further assume that $\varphi \in C^1[0,\infty) \cap C^2(0,\infty)$ and satisfies $\varphi'(t) \sim t\varphi''(t)$, i.e., for some $c_1, c_2 > 0$ it holds

$$c_1 t\varphi''(t) \leq \varphi'(t) \leq c_2 t\varphi''(t) \tag{2.9}$$

uniformly in $t \geq 0$.

Assumption 2 (φ has a p-structure (see [26])). There exist $p \in (1, \infty)$ and $\epsilon \geq 0$ such that $\varphi''(t) \sim (\epsilon + t)^{p-2}$, i.e., for some $c_3, c_4 > 0$ it holds

$$c_3(\epsilon + t)^{p-2} \leq \varphi''(t) \leq c_4(\epsilon + t)^{p-2} \text{ uniformly in } t \geq 0. \tag{2.10}$$

Remark 2.2. *Notice that if Assumptions 1 and 2 are satisfied, then one has uniformly in t*

$$\varphi(t) \sim t\varphi'(t) \sim t^2\varphi''(t) \sim t^2(\epsilon + t)^{p-2}, \tag{2.11}$$

and thus

$$\varphi(t) \lesssim t^2(\epsilon + t)^{p-2} \lesssim \begin{cases} t^p, & 1 < p < 2, \\ \epsilon^{p-2}t^2 + t^p, & p \geq 2. \end{cases} \tag{2.12}$$

Example 1. As a particular example for φ satisfying Assumptions 1 and 2, we will consider

$$\varphi(t) = \frac{(\epsilon + t)^{p-1}t}{p-1} - \frac{(\epsilon + t)^p}{p(p-1)} + \frac{\epsilon^p}{p(p-1)} \tag{2.13}$$

for some $p \in (1, \infty)$ and $\epsilon \geq 0$. In this case we have

$$\varphi'(t) = (\epsilon + t)^{p-2}t \tag{2.14}$$

and

$$\varphi''(t) = \frac{\epsilon + (p-1)t}{(\epsilon + t)^{3-p}}. \tag{2.15}$$

It is easy to see that for any $1 < p < \infty$ and for all $\epsilon \geq 0$, $t \geq 0$ it holds

$$\min\{1, p-1\}(\epsilon + t)^{p-2} \leq \varphi''(t) \leq p(\epsilon + t)^{p-2}. \tag{2.16}$$

Recalling (1.2) we see that in this case the operator $\mathbf{A}$ takes the form

$$\mathbf{A}(\mathbf{a}) = (\epsilon + |\mathbf{a}|)^{p-2}\mathbf{a}. \tag{2.17}$$

Notice that $\epsilon = 0$ corresponds to the standard p-Laplace operator.

Note that by (2.9) and (2.10) $\mathbf{F}$ can be extended by continuity at $\mathbf{a} = \mathbf{0}$, and thus we set $\mathbf{F}(\mathbf{0}) = \mathbf{0}$. For the particular φ' given by (2.13) we have

$$\mathbf{F}(\mathbf{a}) = (\epsilon + |\mathbf{a}|)^{\frac{p-2}{2}}\mathbf{a}. \tag{2.18}$$

Remark 2.3. *By* (2.9) *and* (2.10) *we can estimate*

$$|\mathbf{F}(\mathbf{a})|^2 = \varphi'(|\mathbf{a}|)\,|\mathbf{a}| \leq c_2\,|\mathbf{a}|^2\,\varphi''(|\mathbf{a}|) \leq c_2c_4\,|\mathbf{a}|^2\,(\epsilon + |\mathbf{a}|)^{p-2}. \tag{2.19}$$

Thus, by using (2.12) *we see that* $\mathbf{F}(\mathbf{a}) \in L^2(\Omega)$ *for every* $\mathbf{a} \in [L^p(\Omega)]^d$ *and any* $p > 1$.

Lemma 2.4 (see [21, 19])**.** *Let* $\mathbf{A}$ *be given by* (1.2) *and let* $\mathbf{F}$ *be given by* (1.3)*, where* φ *satisfies Assumption 1. Then the relations*

$$(\mathbf{A}(\mathbf{a}) - \mathbf{A}(\mathbf{b})) \cdot (\mathbf{a} - \mathbf{b}) \sim |\mathbf{F}(\mathbf{a}) - \mathbf{F}(\mathbf{b})|^2 \tag{2.20a}$$

$$\sim |\mathbf{a} - \mathbf{b}|^2 \varphi''(|\mathbf{a}| + |\mathbf{b}|), \tag{2.20b}$$

$$|\mathbf{A}(\mathbf{a}) - \mathbf{A}(\mathbf{b})| \lesssim \varphi''(|\mathbf{a}| + |\mathbf{b}|)\, |\mathbf{a} - \mathbf{b}| \tag{2.20c}$$

hold for all $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$*. More precisely, we will use the following equivalences*

$$c_5\, |\mathbf{F}(\mathbf{a}) - \mathbf{F}(\mathbf{b})|^2 \leq |\mathbf{a} - \mathbf{b}|^2 \varphi''(|\mathbf{a}| + |\mathbf{b}|) \leq c_6\, |\mathbf{F}(\mathbf{a}) - \mathbf{F}(\mathbf{b})|^2 \tag{2.21}$$

with some constants $c_5, c_6 > 0$*. If* $\varphi''(0)$ *does not exist, then the expression in* (2.20b) *is extended by continuity to zero when* $\mathbf{a} = \mathbf{b} = \mathbf{0}$*.*

Remark 2.5. *Notice that Assumption 2 with* $\epsilon > 0$ *combined with the equivalences* (2.20) *imply that* $\mathbf{F}$ *and* $\mathbf{A}$ *are Lipschitz continuous for* $1 < p \leq 2$*. Indeed, for* $\mathbf{F}$ *we obtain*

$$\begin{aligned} |\mathbf{F}(\mathbf{a}) - \mathbf{F}(\mathbf{b})| &\lesssim \sqrt{\varphi''(|\mathbf{a}| + |\mathbf{b}|)}\, |\mathbf{a} - \mathbf{b}| \leq \left(\frac{c_4}{(\epsilon + |\mathbf{a}| + |\mathbf{b}|)^{2-p}} \right)^{\frac{1}{2}} |\mathbf{a} - \mathbf{b}| \\ &\leq \left(\frac{c_4}{\epsilon^{2-p}} \right)^{\frac{1}{2}} |\mathbf{a} - \mathbf{b}| \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^d. \end{aligned} \tag{2.22}$$

Similarly, for $\mathbf{A}$ *we have*

$$|\mathbf{A}(\mathbf{a}) - \mathbf{A}(\mathbf{b})| \lesssim \varphi''(|\mathbf{a}| + |\mathbf{b}|)\, |\mathbf{a} - \mathbf{b}| \lesssim \frac{c_4}{\epsilon^{2-p}}\, |\mathbf{a} - \mathbf{b}| \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^d. \tag{2.23}$$

Now, from Corollary 3.2 in [16] it follows that if $\mathbf{a} \in [W^{1,\bar{p}}(\Omega)]^d$ *for some* $1 \leq \bar{p} \leq \infty$*, then* $\mathbf{F}(\mathbf{a}) \in [W^{1,\bar{p}}(\Omega)]^d$ *and* $\mathbf{A}(\mathbf{a}) \in [W^{1,\bar{p}}(\Omega)]^d$*. In particular, if* $u \in W^{2,\bar{p}}(\Omega)$ *for some* $1 \leq \bar{p} \leq \infty$ *(and thus* $\nabla u \in [W^{1,\bar{p}}(\Omega)]^d$*), then* $\mathbf{F}(\nabla u) \in [W^{1,\bar{p}}(\Omega)]^d$ *and* $\mathbf{A}(\nabla u) \in [W^{1,\bar{p}}(\Omega)]^d$*.*

2.3 Weak formulation

Let φ satisfy Assumption 1 and Assumption 2, and let $\mathbf{A}$ be given by (1.2). The weak formulation of (1.1) reads as follows: Find $u \in W^{1,p}_g(\Omega)$ such that

$$\int_\Omega \mathbf{A}(\nabla u) \cdot \nabla v \, dx = \int_\Omega f v \, dx \text{ for all } v \in W^{1,p}_0(\Omega). \tag{2.24}$$

We will show that the problem (2.24) is equivalent to the minimization problem

$$\begin{aligned} &\text{Find } u \in W^{1,p}_g(\Omega) \text{ such that} \\ &J(u) = \inf_{v \in W^{1,p}_g(\Omega)} J(v), \end{aligned} \tag{2.25}$$

where the functional $J : W^{1,p}_g(\Omega) \to \mathbb{R}$ is defined by

$$J(v) = \int_\Omega \varphi(|\nabla v|)\, dx - \int_\Omega f v \, dx, \tag{2.26}$$

Notice that due to Remark 2.2 and the fact that φ is non-negative, the functional J, indeed, does not take the values $\pm\infty$. The procedure to show the equivalence between (2.24) and (2.25), i.e., show that both have one and the same unique solution, is as follows. First we show existence and uniqueness of a minimizer u of (2.25). Then we show that u actually satisfies (2.24), which essentially means that we have to show the Gateaux-differentiability of J at u. In this case $\langle J'(u), v\rangle = 0$ for all $v \in W_0^{1,p}(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)$. Once we have a solution u of the weak formulation (2.24) we only need to show that it is indeed unique.

We start with the existence and uniqueness of a solution to the variational problem (2.25). Since $W^{1,p}(\Omega)$ is a reflexive Banach space for $1 < p < \infty$ (see, e.g., [1, 12]), existence of a unique minimizer of J can be guaranteed by known theorems from the calculus of variations. In order to apply these theorems we need certain properties for $W_g^{1,p}(\Omega)$ and J which we list below.

Proposition 2.6. *The following assertions hold:*

(1) $W_g^{1,p}(\Omega)$ is convex and closed in $W^{1,p}(\Omega)$ (and thus, weakly closed);

(2) J is sequentially weakly lower semicontinuous[1] *(s.w.l.s.c.), that is*

$$v_n \rightharpoonup v \text{ (weakly) in } W^{1,p}(\Omega) \text{ implies } J(v) \le \liminf_{n\to\infty} J(v_n);$$

(3) J is coercive, that is

$$J(v) \to \infty \text{ whenever } \|v\|_{W^{1,p}(\Omega)} \to \infty.$$

(4) J is strictly convex, i.e., for any $v, w \in W_g^{1,p}(\Omega)$, $v \neq w$, $\lambda \in (0,1)$ it holds

$$J(\lambda v + (1-\lambda)w) < \lambda J(v) + (1-\lambda)J(w),$$

or equivalently, if J is convex and additionally for every pair v, w, the equality $J(\lambda v + (1-\lambda)w) = \lambda J(v) + (1-\lambda)J(w)$ for some $\lambda \in (0,1)$ implies $v = w$.

Proof. For the sake of completeness we proof each of the above assertions.

(1) The convexity and closedness of $W_g^{1,p}(\Omega)$ follow from the linearity and continuity, respectively, of the trace operator γ_p. The convexity and norm closedness of $W_g^{1,p}(\Omega)$ imply that it is also weakly closed (see, e.g., [12, 10]).

(2) To show the second assertion, we first note that the function $\overline{\varphi} : \mathbb{R}^d \to \mathbb{R}^{\ge 0}$ defined by $\overline{\varphi}(\mathbf{a}) := \varphi(|\mathbf{a}|)$ is convex. Indeed, by first using the triangle inequality in $\mathbb{R}^d$ together with the fact that φ is nondecreasing, and then using the convexity of φ, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ and $\lambda \in [0,1]$, we obtain

$$\begin{aligned}\overline{\varphi}(\lambda\mathbf{a} + (1-\lambda)\mathbf{b}) &= \varphi(|\lambda\mathbf{a} + (1-\lambda)\mathbf{b}|) \le \varphi(\lambda\,|\mathbf{a}| + (1-\lambda)\,|\mathbf{b}|) \\ &\le \lambda\varphi(|\mathbf{a}|) + (1-\lambda)\varphi(|\mathbf{b}|) = \lambda\overline{\varphi}(\mathbf{a}) + (1-\lambda)\overline{\varphi}(\mathbf{b}).\end{aligned} \tag{2.27}$$

[1] If J is convex, then it is enough to show that J is lower semicontinuous. Then, the weak lower semicontinuity of J is guaranteed (see, e.g., Corollary 2.2 in [10] or Corollary 3.9 in [12]).

To show that J is s.w.l.s.c., it is enough to notice that it is the sum of two s.w.l.s.c. functionals, $v \mapsto \int_\Omega \varphi(|\nabla v|)\,dx$ and $v \mapsto -\int_\Omega f v dx$. The second one is bounded and linear over $W^{1,p}(\Omega)$, and thus s.w.l.s.c.. For the first one, since $\overline{\varphi}$ is continuous, convex, and satisfies $\overline{\varphi}(\mathbf{a}) \geq 0$ for all $\mathbf{a} \in \mathbb{R}^d$, it follows that the functional

$$v \in W^{1,p}(\Omega) \mapsto \int_\Omega \overline{\varphi}(\nabla v) dx$$

is also s.w.l.s.c. over $W^{1,p}(\Omega)$, see for example Corollary 3.22 in [3].

(3) Now, we show that J is coercive. First, notice that from Assumptions 1 and 2 and (2.10) it follows that

$$\varphi(t) \gtrsim t^2(\epsilon + t)^{p-2}.$$

More precisely, we have

$$\varphi(t) \geq \frac{1}{\Delta_2(\varphi)} c_1 c_3 t^2 (\epsilon + t)^{p-2}. \tag{2.28}$$

By using the expression (2.26) for J together with (2.28) we obtain

$$J(v) \geq \frac{1}{\Delta_2(\varphi)} c_1 c_3 \int_\Omega |\nabla v|^2 (\epsilon + |\nabla v|)^{p-2}\,dx - \int_\Omega f v dx. \tag{2.29}$$

First we consider the case of homogeneous Dirichlet boundary condition $g = 0$, see (1.1b). In this case by using Poincaré's inequality (2.6a) we have $\|\nabla v\|_{L^p(\Omega)} \sim \|v\|_{W^{1,p}(\Omega)}$. Let $p \geq 2$. By first applying Hölder's inequality (2.6b) and then Poincaré's inequality (2.6a) we obtain

$$J(v) \geq \frac{1}{\Delta_2(\varphi)} c_1 c_3 \int_\Omega |\nabla v|^p\,dx - C_P \|f\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} \to \infty \tag{2.30}$$

as $\|v\|_{W^{1,p}(\Omega)} \to \infty$.

Now for $1 < p < 2$, observe that $|\nabla v|^2 \in L^{\frac{p}{2}}(\Omega)$ and $(\epsilon + |\nabla v|)^{p-2} \in L^{\frac{p}{p-2}}(\Omega)$ (note $\left(\frac{p}{2}\right)' = \frac{p}{p-2}$), and thus, we can apply the reverse Hölder inequality (see (2.6c)) to the first integral in (2.29):

$$\begin{aligned} J(v) &\geq \frac{1}{\Delta_2(\varphi)} c_1 c_3 \frac{\|\nabla v\|^2_{L^p(\Omega)}}{\|\epsilon + |\nabla v|\,\|^{2-p}_{L^p(\Omega)}} - C_P \|f\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} \\ &\geq \frac{1}{\Delta_2(\varphi)} c_1 c_3 \frac{\|\nabla v\|^2_{L^p(\Omega)}}{\|\epsilon\|^{2-p}_{L^p(\Omega)} + \|\nabla v\|^{2-p}_{L^p(\Omega)}} - C_P \|f\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} \to \infty \end{aligned} \tag{2.31}$$

as $\|v\|_{W^{1,p}(\Omega)} \to \infty$. In (2.31) we have also used the triangle inequality in $L^p(\Omega)$ together with the inequality (2.6e) with $s = 2 - p \in (0, 1)$.

Now we consider the case of inhomogeneous boundary condition g in (1.1b). Let $u_g \in W^{1,p}(\Omega)$ be such that $\gamma_p(u_g) = g$. Then Poincaré's inequality (2.6a) implies

$$\frac{1}{(C_P^p + 1)^{\frac{1}{p}}} \|v - u_g\|_{W^{1,p}(\Omega)} \leq \|\nabla(v - u_g)\|_{L^p(\Omega)} \leq \|v - u_g\|_{W^{1,p}(\Omega)} \tag{2.32}$$

for all $v \in W_g^{1,p}(\Omega)$. By applying triangle inequality in (2.32) we observe that $\|v\|_{W^{1,p}(\Omega)} \to \infty$ if and only if $\|\nabla v\|_{L^p(\Omega)} \to \infty$. Hence from (2.30) and (2.31) we can conclude that J is also coercive in $W_g^{1,p}(\Omega)$ for both cases $p \geq 2$ and $1 < p < 2$.

(4) For this we first show that $\overline{\varphi}$ is strictly convex on $\mathbb{R}^d$. We note that since φ' is strictly increasing it follows that φ is strictly convex. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, $\lambda \in (0,1)$ and assume that $\overline{\varphi}(\lambda \mathbf{a} + (1-\lambda)\mathbf{b}) = \lambda \overline{\varphi}(\mathbf{a}) + (1-\lambda)\overline{\varphi}(\mathbf{b})$. This means that the two inequality signs in (2.27) are actually equalities. From the second inequality sign being equality, since φ is strictly convex, it follows that $|\mathbf{a}| = |\mathbf{b}|$. From the first inequality sign being equality, since φ is strictly increasing, it follows that $|\lambda \mathbf{a} + (1-\lambda)\mathbf{b}| = \lambda |\mathbf{a}| + (1-\lambda) |\mathbf{b}|$, which is true if and only if $\lambda \mathbf{a} = \alpha(1-\lambda)\mathbf{b}$ for some $\alpha \geq 0$. If $|\mathbf{a}| = |\mathbf{b}| = 0$ then we are done. Let $|\mathbf{a}| = |\mathbf{b}| \neq 0$. Now, it is clear that $\frac{\alpha(1-\lambda)}{\lambda} = 1$, and consequently $\mathbf{a} = \mathbf{b}$.

To see that $v \mapsto \int_\Omega \overline{\varphi}(\nabla v) dx$ is strictly convex, let $v, w \in W_g^{1,p}(\Omega)$ and assume that $J(\lambda v + (1-\lambda)w) = \lambda J(v) + (1-\lambda)J(w)$ for some $\lambda \in (0,1)$. Then it follows

$$\int_\Omega \left[\lambda \overline{\varphi}(\nabla v) + (1-\lambda)\overline{\varphi}(\nabla w) - \overline{\varphi}(\lambda \nabla v + (1-\lambda)\nabla w)\right] dx = 0. \tag{2.33}$$

Since $\overline{\varphi}$ is convex, the integral in (2.33) is nonnegative and therefore it follows that

$$\lambda \overline{\varphi}(\nabla v) + (1-\lambda)\overline{\varphi}(\nabla w) - \overline{\varphi}(\lambda \nabla v + (1-\lambda)\nabla w) = 0 \text{ for a.e. } x \in \Omega. \tag{2.34}$$

From (2.34) and the strict convexity of $\overline{\varphi}$ it follows that $\nabla v = \nabla w$ for a.e. $x \in \Omega$. Since $v - w \in W_0^{1,p}(\Omega)$ we find that $v = w$.

□

The following existence result is well known in the calculus of variations.

Proposition 2.7 (see Proposition 1.2 in [10], Theorem 7.3.7 in [15])**.** *Let V be a reflexive Banach space with norm $\|\cdot\|$ and let $\mathcal{C}$ be a non-empty closed convex subset of V. Let $J : \mathcal{C} \to \mathbb{R}$ be a convex proper sequentially lower semi-continuous functional. Let us assume that either $\mathcal{C}$ is bounded or that J is coercive over $\mathcal{C}$. Then the problem*

$$\textit{Find } u \in \mathcal{C} \textit{ such that } J(u) = \inf_{v \in \mathcal{C}} J(v) \tag{2.35}$$

has at least one solution. It has a unique solution if J is strictly convex.

Theorem 2.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let the function $\varphi : [0,\infty) \to [0,\infty)$ satisfy Assumptions 1 and 2 and let $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$, $f \in L^{p'}(\Omega)$. Then the variational problem* (2.25) *has a unique solution $u \in W_g^{1,p}(\Omega)$.*

Proof. The existence of a unique solution to (2.25) follows directly by combining the assertions in Proposition 2.6 with Proposition 2.7. □

Remark 2.9. *Obviously, the above Theorem 2.8 also holds under weaker assumptions on the data f. For example, $f \in L^{(p^*)'}(\Omega)$ or even $f \in W^{-1,p'}(\Omega)$, the dual space of $W^{1,p}(\Omega)$. Here, p^* denotes the Sobolev conjugate exponent, given by $p^* = \frac{dp}{d-p}$ if $p < d$, $p^* < \infty$ if $p = d$ and $p^* = \infty$ if $p > d$.*

Remark 2.10. *One can also show the coercivity of J by starting from the inequality*

$$(\epsilon + |\nabla v|)^{2-p}\varphi(|\nabla v|) \geq \frac{1}{\Delta_2(\varphi)} c_1 c_3 |\nabla v|^2 ,$$

then raise both sides to the power $\frac{p}{2}$, integrate over Ω and apply the standard Hölder inequality to the left-hand side. This procedure is similar to the one used in [32] in order to show the monotonicity of a mapping associated with the considered nonlinear problem. There, the assumptions on the involved (nonlinear) diffusion coefficient are similar to the one satisfied by the function $t \mapsto \varphi'(t)/t$ in our considerations above.

Theorem 2.11. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let the function $\varphi : [0, \infty) \to [0, \infty)$ satisfy Assumptions 1 and 2 and let $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$, $f \in L^{p'}(\Omega)$. Then problem* (2.24) *has a unique solution $u \in W^{1,p}_g(\Omega)$ which coincides with the unique solution of the variational problem* (2.25)*.*

Proof. First we will show that the solution u of the variational problem (2.25) satisfies the weak formulation (2.24). Then we will show that (2.24) has at most one solution.

By varying the functional J at the minimizer u in directions $v \in W^{1,p}_0(\Omega)$ one finds a necessary condition for u being a minimizer, namely $\langle J'(u), v\rangle = 0$ for all $v \in W^{1,p}_0(\Omega)$. For any $\lambda > 0$ and $v \in W^{1,p}_0(\Omega)$ we have

$$J(u) \leq J(u + \lambda v). \tag{2.36}$$

After dividing both sides of (2.36) by λ and letting $\lambda \to 0^+$ we obtain

$$\lim_{\lambda\to 0^+} \frac{\int_\Omega [\varphi(|\nabla(u + \lambda v)|) - \varphi(|\nabla u|)]\, dx}{\lambda} - \int_\Omega f v dx \geq 0. \tag{2.37}$$

Defining

$$\psi_\lambda(x) := \frac{\varphi(|\nabla(u(x) + \lambda v(x))|) - \varphi(|\nabla u(x)|)}{\lambda}, \tag{2.38}$$

the last inequality can be rewritten in the form

$$\lim_{\lambda\to 0^+} \int_\Omega \psi_\lambda(x) dx - \int_\Omega f v dx \geq 0. \tag{2.39}$$

Our goal is to apply the Lebesgue dominated convergence theorem (LDCT) to the first term in (2.39). By the mean value theorem we find

$$\psi_\lambda(x) = \varphi'(|\nabla u(x) + \Theta(x)\lambda\nabla v(x)|) \frac{\nabla u(x) + \Theta(x)\lambda\nabla v(x)}{|\nabla u(x) + \Theta(x)\lambda\nabla v(x)|} \cdot \nabla v(x), \tag{2.40}$$

where $\Theta(x) \in (0,1)$, and since φ' is continuous, it is clear that

$$\psi_\lambda(x) \to \varphi'(|\nabla u(x)|)\frac{\nabla u(x)}{|\nabla u(x)|} \cdot \nabla v(x) \tag{2.41}$$

for almost every $x \in \Omega$ as $\lambda \to 0^+$. Moreover, for $\lambda < 1$, by first using the triangle inequality in $\mathbb{R}^d$ together with the monotonicity of φ' and then the Cauchy-Schwarz, inequality we find

$$|\psi_\lambda(x)| \leq \varphi'\left(|\nabla u(x)| + |\nabla v(x)|\right)|\nabla v(x)|\,. \tag{2.42}$$

Using (2.9) and (2.10) in (2.42) we find

$$|\psi_\lambda(x)| \leq c_2 c_4 \left(|\nabla u(x)| + |\nabla v(x)|\right)^2 \left(\epsilon + |\nabla u(x)| + |\nabla v(x)|\right)^{p-2}. \tag{2.43}$$

Owing to (2.12) and the fact that $|\nabla u(x)| + |\nabla v(x)| \in L^p(\Omega)$, we see that the right-hand side of (2.43) is in $L^1(\Omega)$. Now having the pointwise convergence (2.41) and the bound (2.43), by applying the LDCT it follows that

$$\lim_{\lambda\to 0^+} \int_\Omega \psi_\lambda(x)dx = \int_\Omega \varphi'(|\nabla u(x)|)\frac{\nabla u(x)}{|\nabla u(x)|} \cdot \nabla v(x)dx, \tag{2.44}$$

and thus

$$\int_\Omega \varphi'(|\nabla u(x)|)\frac{\nabla u(x)}{|\nabla u(x)|} \cdot \nabla v(x)dx - \int_\Omega fvdx \geq 0. \tag{2.45}$$

Recalling the definition (1.2) of the operator $\mathbf{A}$ and the fact that v is an arbitrary element from $W_0^{1,p}(\Omega)$, from (2.45) we conclude that the minimizer u of J satisfies (2.24).

It is left to show that u is indeed the only solution of (2.24). Let $w \in W_g^{1,p}(\Omega)$ is another solution which satisfies (2.24). By subtracting the two associated equations for the solutions u and w and consequently setting $v = u - w$, we find that

$$\int_\Omega \left(\mathbf{A}(\nabla u) - \mathbf{A}(\nabla w)\right) \cdot \left(\nabla u - \nabla w\right) dx = 0. \tag{2.46}$$

From (2.20a) and (2.46) it follows $\mathbf{F}(\nabla u) = \mathbf{F}(\nabla w)$, which due to (1.3) means

$$\left(\varphi'(|\nabla u|)\,|\nabla u|\right)^{\frac{1}{2}} \frac{\nabla u}{|\nabla u|} = \left(\varphi'(|\nabla w|)\,|\nabla w|\right)^{\frac{1}{2}} \frac{\nabla w}{|\nabla w|}. \tag{2.47}$$

By taking absolute values on both sides in (2.47), and using the fact that the function $t \mapsto (\varphi'(t)t)^{\frac{1}{2}}$ is strictly monotone increasing, we obtain $|\nabla u| = |\nabla w|$. Now, from (2.47) it follows that $\nabla u = \nabla w$. Since $u - w \in W_0^{1,p}(\Omega)$ if follows that $u = w$. □

3 Finite element approximation

For simplicity, we assume that Ω is a polyhedral domain. Let $\{\mathcal{T}_h\}_{h>0}$ be a non-degenerate (shape-regular) family of subdivisions of Ω into triangles (or tetrahedra), i.e, $\overline{\Omega} = \cup_{E\in\mathcal{T}_h} E$. Every $\mathcal{T}_h$ is charactirized by the maximum element size $h := \max_{E\in\mathcal{T}_h}\{h_E\}$, where $h_E = \text{diameter(E)}$. We introduce the finite element space $V_h^{(k)}$ defined as

$$V_h^{(k)} := \left\{\phi_h \in C(\overline{\Omega}) : \phi_h|_E \in \mathbb{P}_k(E),\, \forall E \in \mathcal{T}_h\right\}, \tag{3.1}$$

where $\mathbb{P}_k(E)$ is the space of polynomials of degree less than or equal to k on the element E. We introduce the following assumption which will be maintained from now on.

Assumption 3. The boundary datum belongs to the local polynomial space, i.e., there exists $u_{g,h} \in V_h^{(k)}$ such that $\gamma_p(u_{g,h}) = g$.

Under this assumption, we can define the affine space

$$V_{g,h}^{(k)} := \left\{\phi_h \in V_h^{(k)} : \phi_h = g \text{ on } \partial\Omega\right\}. \tag{3.2}$$

Then the finite element discretization of problem (2.24) reads

$$\begin{aligned} &\text{Find } u_h \in V_{g,h}^{(k)} \text{ such that} \\ &\int_\Omega \mathbf{A}(\nabla u_h)\cdot\nabla v_h dx = \int_\Omega f v_h dx \;\text{ for all } v_h \in V_{0,h}^{(k)}. \end{aligned} \tag{3.3}$$

Here, we recall that $\mathbf{A}$ is given in (1.2) and that φ is assumed to satisfy Assumptions 1 and 2. The existence and uniqueness of the finite element solution u_h of (3.3) can be shown by considering the finite dimensional analogue of the variational problem (2.26):

$$\begin{aligned} &\text{Find } u_h \in V_{g,h}^{(k)} \text{ such that} \\ &J(u_h) = \inf_{v_h\in V_{g,h}^{(k)}} J(v_h). \end{aligned} \tag{3.4}$$

We note that $V_{g,h}^{(k)}$ is a closed and convex subset of $W_g^{1,p}(\Omega)$ and that properties (2)-(4) in Proposition 2.6 continue to hold on this subset. Hence, by Proposition 2.7 problem (3.4) has a unique solution u_h. Since J is Gateaux-differentiable at u_h it follows that u_h is also a solution of (3.3). The uniqueness of this solution is shown in a similar way to the one used to prove uniqueness of the continuous problem (2.24).

We cite the following interpolation error estimate, see Theorem (4.4.20) and Corollary (4.4.24) in [31].

Lemma 3.1. *Let $u \in W^{l,\bar{p}}(\Omega)$ with $\bar{p} \geq 1$, $k+1 \geq l \geq 2$. Suppose that $l - d/\bar{p} > 0$ and let I_h^k be the corresponding global interpolation operator in $V_h^{(k)}$. Then, there exists a constant $C_{int,\bar{p},l,\bar{p}} > 0$ independent of h such that the following interpolation estimate holds*

$$\left|u - I_h^k u\right|_{W^{1,\bar{p}}(\Omega)} \leq C_{int,\bar{p},l,\bar{p}} h^{l-1} \left|u\right|_{W^{l,\bar{p}}(\Omega)}. \tag{3.5}$$

If in addition $l - 1 - d/\bar{p} > 0$*, then there exists a constant* $C_{int,\infty,l,\bar{p}} > 0$ *such that*

$$\left|u - I_h^k u\right|_{W^{1,\infty}(\Omega)} \leq C_{int,\infty,l,\bar{p}} h^{l-1-d/\bar{p}} \left|u\right|_{W^{l,\bar{p}}(\Omega)}. \tag{3.6}$$

Furthermore, we recall the near-best approximation result from [21] with respect to $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$. Note that by Remark 2.3 this quantity is well defined.

Proposition 3.2. *Let* $u \in W_g^{1,p}(\Omega)$ *be the solution of* (2.24) *and let* $u_h \in V_{g,h}^{(k)}$ *be the Galerkin approximation of* u *defined by* (3.3)*. Then*

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \leq c_7 \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)} \text{ for all } v_h \in V_{g,h}^{(k)}. \tag{3.7}$$

Furthermore, if $\mathbf{F}(\nabla u) \in \left[W^{1,2}(\Omega)\right]^d$*, then*

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \Pi_h^k u)\|_{L^2(\Omega)} \leq Ch \|\nabla \mathbf{F}(\nabla u)\|_{L^2(\Omega)}, \tag{3.8}$$

where $\Pi_h^k : W^{1,p}(\Omega) \to V_h^{(k)}$ *is for example the Scott-Zhang interpolation operator in* $V_h^{(k)}$*. In this case, by* (3.7) *and* (3.8) *we have*

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \leq c_7 \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \Pi_h^k u)\|_{L^2(\Omega)} \leq Ch \|\nabla \mathbf{F}(\nabla u)\|_{L^2(\Omega)}. \tag{3.9}$$

Remark 3.3. *Notice that the statements in Proposition 3.2 are valid for any fixed* $1 < p < \infty$ *and* $\epsilon \geq 0$ *in Assumption 2.*

3.1 Near-best approximation results in the $W^{1,p}$ seminorm

Let φ satisfy Assumption 1 and Assumption 2 for some $p > 1$. We distinguish two main cases for p: $p \geq 2$ and $1 < p < 2$. For each case, we proceed in three steps. In the first step we derive bounds for the $W^{1,p}$ seminorm of u, which depend only on the problem data. Analogous bounds are given for the corresponding FE approximation u_h. The second step consists of the estimation of the discretization error $|u - u_h|_{W^{1,p}(\Omega)}$ in terms of the discretization error $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$ and the approximation error $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)}$ in terms of the approximation error $|u - v_h|_{W^{1,p}(\Omega)}$. Finally, in the third step, we combine the discretization and approximation estimates from the previous steps with the near-best approximation result (3.7).

3.1.1 The case $p \geq 2$

Homogeneous Dirichlet boundary condition: Let us first consider the case $g = 0$ on $\partial\Omega$. Setting $v = u$ in (2.24) and using (2.9) and (2.10) together with Hölder's inequality (2.6b) and Poincaré's inequality (2.6), for all $\epsilon \geq 0$ we obtain

$$c_1 c_3 \int_\Omega |\nabla u|^p \, dx \leq c_1 c_3 \int_\Omega (\epsilon + |\nabla u|)^{p-2} |\nabla u|^2 \, dx \leq C_P \|f\|_{L^{p'}(\Omega)} \|\nabla u\|_{L^p(\Omega)}, \tag{3.10}$$

where $p' := p/(p-1)$ is the Hölder conjugate of p. From (3.10) it follows that

$$\|\nabla u\|_{L^p(\Omega)} \leq \left(\frac{C_P}{c_1 c_3}\|f\|_{L^{p'}(\Omega)}\right)^{\frac{1}{p-1}} =: M_1^0(\Omega, f, p). \tag{3.11}$$

Similarly, setting $v_h = u_h$ in (3.3) we obtain

$$\|\nabla u_h\|_{L^p(\Omega)} \leq M_1^0(\Omega, f, p). \tag{3.12}$$

Nonhomogeneous Dirichlet boundary condition: Now, consider the case where g is not identically zero on $\partial\Omega$. From Assumption 3 there is a function $u_{g,h} \in V_{g,h}^{(k)}$ such that $\gamma_p(u_{g,h}) = g$. By setting $v = u - u_{g,h} \in W_0^{1,p}(\Omega)$ in the weak formulation (2.24) and recalling the definition (1.2) of $\mathbf{A}$ we have

$$\int_\Omega \varphi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla u_{g,h})dx = \int_\Omega f(u - u_{g,h})dx. \tag{3.13}$$

By rearranging the terms in (3.13) and applying Hölder's and Poincaré's inequalities (2.6b) and (2.6a) we obtain

$$\int_\Omega \varphi'(|\nabla u|)\,|\nabla u|\,dx \leq C_P\|f\|_{L^{p'}(\Omega)}\left(\|\nabla u\|_{L^p(\Omega)} + \|\nabla u_{g,h}\|_{L^p(\Omega)}\right) + \int_\Omega \varphi'(|\nabla u|)\,|\nabla u_{g,h}|\,dx. \tag{3.14}$$

Now, we estimate appropriately the relevant terms in (3.14). From (2.9) and (2.10) it follows that

$$\int_\Omega \varphi'(|\nabla u|)\,|\nabla u|\,dx \geq c_1 c_3 \int_\Omega |\nabla u|^2\,(\epsilon + |\nabla u|)^{p-2}\,dx \geq c_1 c_3 \|\nabla u\|_{L^p(\Omega)}^p. \tag{3.15}$$

Next, by applying Young's inequality (2.6d) with $\delta_1 > 0$ we find

$$C_P\|f\|_{L^{p'}(\Omega)}\|\nabla u\|_{L^p(\Omega)} \leq \frac{\delta_1\|\nabla u\|_{L^p(\Omega)}^p}{p} + \frac{\left(C_P\|f\|_{L^{p'}(\Omega)}\right)^{p'}}{\delta_1^{\frac{p'}{p}}\,p'}. \tag{3.16}$$

Finally, we estiamte the last term in (3.14) from above. By using (2.9), (2.10), and then (2.6e) with $s = p-2 > 0$ and $C(s) = 2^s$ we find for $t \geq 0$

$$0 \leq \varphi'(t) \leq c_2 c_4 t(\epsilon + t)^{p-2} \leq 2^{p-2}c_2 c_4 t\epsilon^{p-2} + 2^{p-2}c_2 c_4 t^{p-1}. \tag{3.17}$$

If $t \leq 1$, then (3.17) implies

$$\varphi'(t) \leq 2^{p-2}c_2 c_4 \epsilon^{p-2} + 2^{p-2}c_2 c_4 =: m_1. \tag{3.18}$$

If, on the other hand, $t > 1$ then (3.17) implies

$$\varphi'(t) \leq 2^{p-2}c_2 c_4\left(\epsilon^{p-2} + 1\right)t^{p-1} = m_1 t^{p-1}. \tag{3.19}$$

Together (3.18) and (3.19) imply

$$\varphi'(t) \le m_1 + m_1 t^{p-1} \text{ for all } t \ge 0. \tag{3.20}$$

By sequentially applying Hölder's inequality (2.6b), (3.20), triangle inequality in $L^{p'}(\Omega)$, and then Young's inequality with $\delta_2 > 0$ we estimate

$$\begin{aligned}\int\limits_\Omega \varphi'(|\nabla u|)\,|\nabla u_{g,h}|\,dx &\le \|\varphi'(|\nabla u|)\|_{L^{p'}(\Omega)} \|\nabla u_{g,h}\|_{L^p(\Omega)} \\ &\le \left(m_1\,|\Omega|^{\frac{1}{p'}} + m_1 \|\,|\nabla u|^{p-1}\,\|_{L^q(\Omega)}\right) \|\nabla u_{g,h}\|_{L^p(\Omega)} \\ &\le m_1\,|\Omega|^{\frac{1}{p'}}\,\|\nabla u_{g,h}\|_{L^p(\Omega)} + \frac{\delta_2 \|\nabla u\|^p_{L^p(\Omega)}}{p'} + \frac{\left(m_1 \|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^p}{\delta_2^{\frac{p}{p'}}\,p},\end{aligned} \tag{3.21}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. By combining (3.14), (3.15), (3.16), and (3.21) we obtain for $\delta_1 = (c_1 c_3 p)/4$ and $\delta_2 = (c_1 c_3 p')/4$

$$\begin{aligned}\frac{c_1 c_3}{2} \|\nabla u\|^p_{L^p(\Omega)} \le & C_P \|f\|_{L^{p'}(\Omega)} \|\nabla u_{g,h}\|_{L^p(\Omega)} + \frac{4^{\frac{p'}{p}} \left(C_P \|f\|_{L^{p'}(\Omega)}\right)^{p'}}{(c_1 c_3 p)^{\frac{p'}{p}}\,p'} \\ & + m_1\,|\Omega|^{\frac{1}{p'}}\,\|\nabla u_{g,h}\|_{L^p(\Omega)} + \frac{4^{\frac{p}{p'}} \left(m_1 \|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^p}{(c_1 c_3 p')^{\frac{p}{p'}}\,p}.\end{aligned} \tag{3.22}$$

From (3.22) we find the estimate

$$\|\nabla u\|_{L^p(\Omega)} \le M_1^g(\Omega, f, u_{g,h}, p, \epsilon), \tag{3.23}$$

where the constant M_1^g is given by

$$\begin{aligned}M_1^g(\Omega, f, u_{g,h}, p, \epsilon) := \left(\frac{2}{c_1 c_3}\right)^{\frac{1}{p}} \Bigg(& C_P \|f\|_{L^{p'}(\Omega)} \|\nabla u_{g,h}\|_{L^p(\Omega)} + \frac{4^{\frac{p'}{p}} \left(C_P \|f\|_{L^{p'}(\Omega)}\right)^{p'}}{(c_1 c_3 p)^{\frac{p'}{p}}\,p'} \\ & + m_1\,|\Omega|^{\frac{1}{p'}}\,\|\nabla u_{g,h}\|_{L^p(\Omega)} + \frac{4^{\frac{p}{p'}} \left(m_1 \|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^p}{(c_1 c_3 p')^{\frac{p}{p'}}\,p} \Bigg)^{\frac{1}{p}}.\end{aligned} \tag{3.24}$$

In a similar way, by testing (3.3) with $v_h := u_h - u_{g,h} \in V_{0,h}^{(k)}$ one can obtain the estimate

$$\|\nabla u_h\|_{L^p(\Omega)} \le M_1^g(\Omega, f, u_{g,h}, p, \epsilon). \tag{3.25}$$

Next, we derive an estimate for the discretization error $|u - u_h|_{W^{1,p}(\Omega)}$ in terms of the discretization error $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$. Using (2.21) and (2.10) we have that

$$
\begin{aligned}
&|u-u_h|^p_{W^{1,p}(\Omega)} = \int_\Omega |\nabla u - \nabla u_h|^p \, dx \\
=&\int_\Omega |\nabla u - \nabla u_h|^2 \varphi''(|\nabla u| + |\nabla u_h|) \frac{|\nabla u - \nabla u_h|^{p-2}}{\varphi''(|\nabla u| + |\nabla u_h|)} dx \\
\leq& c_6 \int_\Omega |\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)|^2 \frac{(\epsilon + |\nabla u| + |\nabla u_h|)^{p-2}}{c_3 (\epsilon + |\nabla u| + |\nabla u_h|)^{p-2}} dx \\
=&\frac{c_6}{c_3} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.26}
$$

On the other hand, by first using (2.21) and Hölder's inequality (2.6b) in $L^{\frac{p}{2}}(\Omega)$ and $L^{\left(\frac{p}{2}\right)'}(\Omega) = L^{\frac{p}{p-2}}(\Omega)$, and then (2.10) together with the triangle inequality in $L^p(\Omega)$ and the estimate (3.10), for all v_h satisfying $\|\nabla v_h\|_{L^p(\Omega)} \leq \widetilde{M}_1^g$ for some $\widetilde{M}_1^g \geq M_1^g(\Omega, f, u_{g,h}, p, \epsilon)$, we estimate

$$
\begin{aligned}
&\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|^2_{L^2(\Omega)} \leq \frac{1}{c_5} \int_\Omega |\nabla u - \nabla v_h|^2 \varphi''(|\nabla u| + |\nabla v_h|) \, dx \\
\leq& \frac{1}{c_5} \left(\int_\Omega |\nabla u - \nabla v_h|^p \, dx \right)^{\frac{2}{p}} \left(\int_\Omega c_4^{\frac{p}{p-2}} (\epsilon + |\nabla u| + |\nabla v_h|)^p \, dx \right)^{\frac{p-2}{p}} \\
\leq& \frac{c_4}{c_5} \left(|\Omega|^{\frac{1}{p}} \epsilon + \| \, |\nabla u| \, \|_{L^p(\Omega)} + \| \, |\nabla v_h| \, \|_{L^p(\Omega)} \right)^{p-2} |u - v_h|^2_{W^{1,p}(\Omega)} \\
\leq& \frac{c_4}{c_5} \left(|\Omega|^{\frac{1}{p}} \epsilon + M_1^g + \widetilde{M}_1^g \right)^{p-2} |u - v_h|^2_{W^{1,p}(\Omega)} .
\end{aligned}
\tag{3.27}
$$

Finally, by combining (3.7) with (3.26) and (3.27) we arrive at a near-best approximation result in terms of the seminorm $|\cdot|_{W^{1,p}(\Omega)}$: For all $v_h \in V_{g,h}^{(k)}$ satisfying $\|\nabla v_h\|_{L^p(\Omega)} \leq \widetilde{M}_1^g$ it holds

$$
\begin{aligned}
|u - u_h|_{W^{1,p}(\Omega)} &\leq \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|^{\frac{2}{p}}_{L^2(\Omega)} \\
&\leq \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} c_7^{\frac{2}{p}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|^{\frac{2}{p}}_{L^2(\Omega)} \leq C_1 |u - v_h|^{\frac{2}{p}}_{W^{1,p}(\Omega)} ,
\end{aligned}
\tag{3.28}
$$

where

$$
C_1(\Omega, f, \epsilon, p, \widetilde{M}_1^g) := \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} c_7^{\frac{2}{p}} \left(\frac{c_4}{c_5}\right)^{\frac{1}{p}} \left(|\Omega|^{\frac{1}{p}} \epsilon + M_1^g + \widetilde{M}_1^g \right)^{\frac{p-2}{p}} . \tag{3.29}
$$

Proposition 3.4. *Let φ satisfy Assumption 1 and Assumption 2 with $p > 2$. Let $u \in W_g^{1,p}(\Omega)$ be the solution of (2.24) and let $u_h \in V_{g,h}^{(k)}$ be the Galerkin approximation* *of u defined by* (3.3)*. Then if we choose $\widetilde{M}_1^g = 2M_1^g + \|\nabla u_{g,h}\|_{L^p(\Omega)}$ it holds*

$$
\begin{aligned}
|u - u_h|_{W^{1,p}(\Omega)} &\le \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}^{\frac{2}{p}} \\
&\le \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} c_7^{\frac{2}{p}} \inf_{v_h \in V_{g,h}^{(k)}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)}^{\frac{2}{p}} \\
&\le C_1\left(\Omega, f, \epsilon, p, 2M_1^g + \|\nabla u_{g,h}\|_{L^p(\Omega)}\right) \inf_{v_h \in V_{g,h}^{(k)}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}},
\end{aligned}
\tag{3.30}
$$

where $M_1^g(\Omega, f, u_{g,h}, p, \epsilon)$ is defined in (3.24)*, C_1 is defined in* (3.29)*, and $u_{g,h}$ is a fixed function from $V_{g,h}^{(k)}$ (see Assumption 3).*

Proof. First, observe that we can take the infimum over all functions $v_h \in V_{g,h}^{(k)}$ in the first two inequalities in (3.28) since those hold regardless of whether $\|\nabla v_h\|_{L^p(\Omega)} \le \widetilde{M}_1^g$ is satisfiesd (they follow from (3.26) and (3.7)). Therefore, it is clear that

$$
\begin{aligned}
&|u - u_h|_{W^{1,p}(\Omega)} \le \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}^{\frac{2}{p}} \\
&\le \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} c_7^{\frac{2}{p}} \inf_{v_h \in V_{g,h}^{(k)}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)}^{\frac{2}{p}} \le \left(\frac{c_6}{c_3}\right)^{\frac{1}{p}} c_7^{\frac{2}{p}} \inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} \le \widetilde{M}_1^g}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)}^{\frac{2}{p}} \\
&\le C_1\left(\Omega, f, \epsilon, p, 2M_1^g + \|\nabla u_{g,h}\|_{L^p(\Omega)}\right) \inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} \le \widetilde{M}_1^g}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}}.
\end{aligned}
\tag{3.31}
$$

Now, it is left to show that

$$
\inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} \le \widetilde{M}_1^g}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}} = \inf_{v_h \in V_{g,h}^{(k)}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}}
\tag{3.32}
$$

holds with the above defined $\widetilde{M}_1^g$. Obviously, we have the inequality

$$
\inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} \le \widetilde{M}_1^g}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}} \ge \inf_{v_h \in V_{g,h}^{(k)}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}}.
\tag{3.33}
$$

Now, we want to show the opposite inequality to (3.33). First, observe that the following identity holds:

$$
\inf_{v_h \in V_{g,h}^{(k)}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}} = \min \left\{ \inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} \le \widetilde{M}_1^g}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}}, \inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} > \widetilde{M}_1^g}} |u - v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}} \right\}.
\tag{3.34}
$$

By using the triangle inequality in $L^p(\Omega)$ we obtain

$$\|\nabla(u-v_h)\|_{L^p(\Omega)}^{\frac{2}{p}} \geq \left(\|\nabla v_h\|_{L^p(\Omega)} - \|\nabla u\|_{L^p(\Omega)}\right)^{\frac{2}{p}} > \left(M_1^g + \|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^{\frac{2}{p}} \quad (3.35)$$

for all functions v_h such that $\|\nabla v_h\|_{L^p(\Omega)} > \widetilde{M}_1^g$. On the other hand, again by applying the triangle inequality in $L^p(\Omega)$ we find

$$\inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} \leq \widetilde{M}_1^g}} |u-v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}} \leq |u-u_{g,h}|_{W^{1,p}(\Omega)}^{\frac{2}{p}} \leq \left(M_1^g + \|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^{\frac{2}{p}} \quad (3.36)$$

Taking the infimum in (3.35) over all functions v_h with $\|\nabla v_h\|_{L^p(\Omega)} > \widetilde{M}_1^g$ and combining with (3.36) we find

$$\inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} \leq \widetilde{M}_1^g}} |u-v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}} < \inf_{\substack{v_h \in V_{g,h}^{(k)} \\ \|\nabla v_h\|_{L^p(\Omega)} > \widetilde{M}_1^g}} |u-v_h|_{W^{1,p}(\Omega)}^{\frac{2}{p}}. \quad (3.37)$$

Now, (3.37) and (3.34) imply the equality (3.32). □

Remark 3.5. *Note that in the proof of* (3.27) *are only used Assumption 1 (which ensures that the statement in Lemma 2.4 holds) and the upper bound in* (2.10) *together with the bounds* (3.23) *and* (3.25) *on* $\|\nabla u\|_{L^p(\Omega)}$ *and* $\|\nabla u_h\|_{L^p(\Omega)}$*, respectively. In order to show the bounds on* $\|\nabla u\|_{L^p(\Omega)}$ *and* $\|\nabla u_h\|_{L^p(\Omega)}$ *we have used also the lower bound in* (2.10)*. However, this can be avoided by letting the constant* C_1 *depend on* $\|\nabla u\|_{L^p(\Omega)}$ *and on the constant* $\widetilde{M}_1^g \geq \|\nabla u\|_{L^p(\Omega)}$*. Indeed, from the near-best approximation result* (3.7) *and the estimate* (3.27) *one obtains that for all* v_h *satisfying* $\|\nabla v_h\|_{L^p(\Omega)} \leq \widetilde{M}_1^g$ *for some* $\widetilde{M}_1^g \geq \|\nabla u\|_{L^p(\Omega)}$ *it holds*

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \leq c_7 \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)} \leq C_1' \, |u-v_h|_{W^{1,p}(\Omega)}, \quad (3.38)$$

where

$$C_1' := \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}} \left(|\Omega|^{\frac{1}{p}} \epsilon + \|\nabla u\|_{L^p(\Omega)} + \widetilde{M}_1^g\right)^{\frac{p-2}{2}}. \quad (3.39)$$

Then, repeating the proof of Proposition 3.4 with $\widetilde{M}_1^g = 2\|\nabla u\|_{L^p(\Omega)} + \|\nabla u_{g,h}\|_{L^p(\Omega)}$ *we see that* (3.32) *still holds. Therefore, we obtain the near-best approximation result*

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \leq c_7 \inf_{v_h \in V_{g,h}^{(k)}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)} \leq C_1' \inf_{v_h \in V_{g,h}^{(k)}} |u-v_h|_{W^{1,p}(\Omega)}, \quad (3.40)$$

where

$$C_1' := \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}} \left(|\Omega|^{\frac{1}{p}} \epsilon + 3\|\nabla u\|_{L^p(\Omega)} + \|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^{\frac{p-2}{2}}. \quad (3.41)$$

3.1.2 The case $1 < p < 2$

Homogeneous Dirichlet boundary condition: Let us first consider the case $g = 0$ on $\partial\Omega$. Setting $v = u$ in (2.24) and using (2.9) and (2.10) together with Hölder's inequality (2.6b) and Poincaré's inequality (2.6a), for all $\epsilon \geq 0$ we obtain

$$c_1 c_3 \int_\Omega (\epsilon + |\nabla u|)^{p-2} |\nabla u|^2 \, dx \leq \int_\Omega \varphi'(|\nabla u|) \, |\nabla u| \, dx \leq C_P \|f\|_{L^{p'}(\Omega)} \|\nabla u\|_{L^p(\Omega)}. \tag{3.42}$$

Next, using the reverse Hölder's inequality (2.6c) on the left-hand side of (3.42) together with the triangle inequality in $L^p(\Omega)$ and the inequality (2.6e) with $s = 2 - p \in (0,1)$ we find

$$c_1 c_3 \frac{\|\nabla u\|^2_{L^p(\Omega)}}{\|\epsilon\|^{2-p}_{L^p(\Omega)} + \|\nabla u\|^{2-p}_{L^p(\Omega)}} \leq C_P \|f\|_{L^{p'}(\Omega)} \|\nabla u\|_{L^p(\Omega)}, \tag{3.43}$$

from where it follows (if $\|\nabla u\|_{L^p(\Omega)} = 0$, then the inequality below still holds)

$$\|\nabla u\|_{L^p(\Omega)} \leq \frac{C_P \|f\|_{L^{p'}(\Omega)}}{c_1 c_3} \left(\|\epsilon\|^{2-p}_{L^p(\Omega)} + \|\nabla u\|^{2-p}_{L^p(\Omega)} \right). \tag{3.44}$$

Considering the two possible cases (i) $\|\nabla u\|_{L^p(\Omega)} \leq \|\epsilon\|_{L^p(\Omega)}$ or (ii) $\|\nabla u\|_{L^p(\Omega)} > \|\epsilon\|_{L^p(\Omega)}$, from (3.44) we can derive the energy estimate

$$\|\nabla u\|_{L^p(\Omega)} \leq \max \left\{ \|\epsilon\|_{L^p(\Omega)}, \left(\frac{2 C_P \|f\|_{L^{p'}(\Omega)}}{c_1 c_3} \right)^{\frac{1}{p-1}} \right\} =: M_2^0 (\Omega, f, p, \epsilon). \tag{3.45}$$

Similarly, by taking $v_h = u_h$ in (3.3) and repeating the steps above we find

$$\|\nabla u_h\|_{L^p(\Omega)} \leq M_2^0 (\Omega, f, p, \epsilon). \tag{3.46}$$

Nonhomogeneous Dirichlet boundary condtion: In the case where g is not identically zero on $\partial\Omega$, by setting $v = u - u_{g,h} \in W_0^{1,p}(\Omega)$ in the weak formulation (2.24) we can obtain (3.14) in a similar way to the case $p \geq 2$. We estimate appropriately the terms containing the solution u in (3.14). As in (3.43), we find with the help of the reverse Hölder's inequality (2.6c)

$$c_1 c_3 \frac{\|\nabla u\|^2_{L^p(\Omega)}}{\|\epsilon\|^{2-p}_{L^p(\Omega)} + \|\nabla u\|^{2-p}_{L^p(\Omega)}} \leq \int_\Omega \varphi'(|\nabla u|) \, |\nabla u| \, dx. \tag{3.47}$$

By applying Young's inequality (2.6d) with $\delta_3 > 0$ we find

$$C_P \|f\|_{L^{p'}(\Omega)} \|\nabla u\|_{L^p(\Omega)} \leq \frac{\delta_3 \|\nabla u\|^p_{L^p(\Omega)}}{p} + \frac{\left(C_P \|f\|_{L^{p'}(\Omega)} \right)^{p'}}{\delta_3^{\frac{p'}{p}} p'}. \tag{3.48}$$

Next, by (2.9) and (2.10) we have

$$\varphi'(t) \leq c_2 c_4 t (\epsilon + t)^{p-2} \leq c_2 c_4 t^{p-1} \quad \text{for all } t \geq 0. \tag{3.49}$$

Therefore, for the last term in (3.14) we obtain by using Hölder's inequality (2.6b) and Young's inequality (2.6d) for some $\delta_4 > 0$

$$\int_\Omega \varphi'(|\nabla u|)\,|\nabla u_{g,h}|\,dx \le \|\varphi'(|\nabla u|)\|_{L^{p'}(\Omega)}\|\nabla u_{g,h}\|_{L^p(\Omega)} \le c_2c_4\|\,|\nabla u|^{p-1}\,\|_{L^{p'}(\Omega)}\|\nabla u_{g,h}\|_{L^p(\Omega)}$$
$$\le \frac{\delta_4\|\nabla u\|^p_{L^p(\Omega)}}{p'} + \frac{\left(c_2c_4\|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^p}{\delta_4^{\frac{p}{p'}}\,p}. \tag{3.50}$$

Now, we combine the above obtained estimates. If $\|\nabla u\|_{L^p(\Omega)} \le \|\epsilon\|_{L^p(\Omega)}$, then we do not have to do anything more. If, on the other hand, $\|\nabla u\|_{L^p(\Omega)} > \|\epsilon\|_{L^p(\Omega)}$, then from (3.47) it follows

$$\frac{c_1c_3}{2}\|\nabla u\|^p_{L^p(\Omega)} \le \int_\Omega \varphi'(|\nabla u|)\,|\nabla u|\,dx. \tag{3.51}$$

Combining (3.51), (3.14), (3.48) for $\delta_3 = (c_1c_3p)/8$, and (3.50) for $\delta_4 = (c_1c_3p')/8$ we estimate

$$\frac{c_1c_3}{4}\|\nabla u\|^p_{L^p(\Omega)} \le C_P\|f\|_{L^{p'}(\Omega)}\|\nabla u_{g,h}\|_{L^p(\Omega)} + \frac{\left(C_P\|f\|_{L^{p'}(\Omega)}\right)^{p'}}{\delta_3^{\frac{p'}{p}}\,p'} + \frac{\left(c_2c_4\|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^p}{\delta_4^{\frac{p}{p'}}\,p}. \tag{3.52}$$

From (3.52) follows that

$$\|\nabla u\|_{L^p(\Omega)} \le M_2^g(\Omega, f, u_{g,h}, p, \epsilon), \tag{3.53}$$

where

$$M_2^g(\Omega, f, u_{g,h}, p, \epsilon) :=$$
$$\left(\frac{4}{c_1c_3}\right)^{\frac{1}{p}}\left(C_P\|f\|_{L^{p'}(\Omega)}\|\nabla u_{g,h}\|_{L^p(\Omega)} + \frac{8^{\frac{p'}{p}}\left(C_P\|f\|_{L^{p'}(\Omega)}\right)^{p'}}{(c_1c_3p)^{\frac{p'}{p}}\,p'} + \frac{8^{\frac{p}{p'}}\left(c_2c_4\|\nabla u_{g,h}\|_{L^p(\Omega)}\right)^p}{(c_1c_3p')^{\frac{p}{p'}}\,p}\right)^{\frac{1}{p}}. \tag{3.54}$$

In a similar way, by testing (3.3) with $v_h := u_h - u_{g,h} \in V_{0,h}^{(k)}$ one can obtain the estimate

$$\|\nabla u_h\|_{L^p(\Omega)} \le M_2^g(\Omega, f, u_{g,h}, p, \epsilon). \tag{3.55}$$

Next we derive an upper bound for the discretization error $|u - u_h|_{W^{1,p}}$ in terms of the discretization error $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$. Applying sequentially (2.21) and (2.10), Hölder's inequality (2.6b) in $L^{\frac{2}{p}}(\Omega)$ and $L^{\left(\frac{2}{p}\right)'}(\Omega)$ (note that $\left(\frac{2}{p}\right)' = \frac{2}{2-p}$), we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla u - \nabla u_h|^p \, dx \\
=&\int_\Omega |\nabla u - \nabla u_h|^p \, \varphi''\left(|\nabla u| + |\nabla u_h|\right)^{\frac{p}{2}} \frac{1}{\varphi''\left(|\nabla u| + |\nabla u_h|\right)^{\frac{p}{2}}} dx \\
\leq &\left(\frac{c_6}{c_3}\right)^{\frac{p}{2}} \int_\Omega |\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)|^p \frac{1}{\left(\epsilon + |\nabla u| + |\nabla u_h|\right)^{\frac{p(p-2)}{2}}} dx \\
\leq &\left(\frac{c_6}{c_3}\right)^{\frac{p}{2}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|^p_{L^2(\Omega)} \left(\int_\Omega \left(\epsilon + |\nabla u| + |\nabla u_h|\right)^p dx\right)^{\frac{2-p}{2}} \\
\leq &N_1 \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|^p_{L^2(\Omega)}.
\end{aligned}
\tag{3.56}
$$

In the last step of (3.56), we have used the triangle inequality in $L^p(\Omega)$ and the bounds (3.53) and (3.55). Thus, the constant N_1 is given by

$$
N_1\left(\Omega, f, u_{g,h}, p, \epsilon\right) := \left(|\Omega|^{\frac{1}{p}} \epsilon + 2M_2^g\right)^{\frac{p(2-p)}{2}} \left(\frac{c_6}{c_3}\right)^{\frac{p}{2}}. \tag{3.57}
$$

From here we arrive at

$$
|u - u_h|_{W^{1,p}(\Omega)} \leq \ N_1^{\frac{1}{p}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}. \tag{3.58}
$$

By combining (3.58) with (3.7), for all $v_h \in V_{g,h}^{(k)}$ it holds

$$
|u - u_h|_{W^{1,p}(\Omega)} \leq c_7 N_1^{\frac{1}{p}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)}. \tag{3.59}
$$

We now estimate the right-hand side of (3.59) in terms of $|u - v_h|_{W^{1,p}(\Omega)}$. Using (2.21) and (2.10) we have

$$
\begin{aligned}
&\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|^2_{L^2(\Omega)} \leq \frac{1}{c_5} \int_\Omega |\nabla u - \nabla v_h|^2 \varphi''\left(|\nabla u| + |\nabla v_h|\right) dx \\
=&\frac{1}{c_5} \int_\Omega |\nabla u - \nabla v_h|^p \varphi''\left(|\nabla u| + |\nabla v_h|\right) |\nabla u - \nabla v_h|^{2-p} dx \\
\leq&\frac{c_4}{c_5} \int_\Omega |\nabla u - \nabla v_h|^p \frac{|\nabla u - \nabla v_h|^{2-p}}{\left(\epsilon + |\nabla u| + |\nabla v_h|\right)^{2-p}} dx \\
\leq&\frac{c_4}{c_5} \int_\Omega |\nabla u - \nabla v_h|^p \frac{\left(\epsilon + |\nabla u| + |\nabla v_h|\right)^{2-p}}{\left(\epsilon + |\nabla u| + |\nabla v_h|\right)^{2-p}} dx = \frac{c_4}{c_5} |u - v_h|^p_{W^{1,p}(\Omega)}.
\end{aligned}
\tag{3.60}
$$

In this way, we arrive at a near-best approximation result in terms of the semi-norm $|\cdot|_{W^{1,p}(\Omega)}$.

Proposition 3.6. *Let φ satisfy Assumption 1 and Assumption 2 with $1 < p < 2$. Let $u \in W_g^{1,p}(\Omega)$ be the solution of (2.24) and let $u_h \in V_{g,h}^{(k)}$ be the finite element solution* *defined by* (3.3)*. Then the estimate*

$$\frac{1}{N_1^{\frac{1}{p}}}|u-u_h|_{W^{1,p}(\Omega)} \le \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \le c_7\|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla v_h)\|_{L^2(\Omega)}$$
$$\le c_7\left(\frac{c_4}{c_5}\right)^{\frac{1}{2}}|u-v_h|^{\frac{p}{2}}_{W^{1,p}(\Omega)} \quad \text{for all } v_h \in V^{(k)}_{g,h} \tag{3.61}$$

holds with $N_1(\Omega, f, u_{g,h}, p, \epsilon)$ *defined in* (3.57)*.*

Proof. By combining (3.58), the near-best approximation result (3.7), and (3.60) we obtain (3.61). □

Remark 3.7. *Note that in the proof of* (3.60) *only the upper bound in* (2.10) *is used. Therefore, if existence and uniqueness of the solutions* u *and* u_h *of* (2.24) *and* (3.3)*, respectively, are granted to us, then the last two inequalities in* (3.61) *hold by requiring only Assumption 1 (which ensures that the statement in Lemma 2.4 holds) and the upper bound in* (2.10)*.*

Remark 3.8. *Notice that the near-best approximation result* (3.30) *was proved in [32] under similar assumptions on the data and the nonlinear differential operator by employing an approach from [4]. On the other hand, the analogous result in the case* $p \ge 2$ *that is proved in [32] is slightly weaker than* (3.30)*. More precisely, in [32] it is proved that*

$$|u-u_h|_{W^{1,p}(\Omega)} \le C\left|u - I_h^k u\right|^{\frac{2}{p}}_{W^{1,p}(\Omega)} \quad \text{for} \quad p \ge 2,$$

where the constant C *may depend on* $\|\nabla I_h^k u\|_{L^p(\Omega)}$*.*

Remark 3.9. *From* (3.56) *and* (3.60) *we can deduce the following pointwise inequalities which are valid for any* $v_h \in V^{(k)}_{g,h}$*:*

$$|\nabla u - \nabla v_h|^p \le \left(\frac{c_6}{c_3}\right)^{\frac{p}{2}}|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla v_h)|^p(\epsilon + |\nabla u| + |\nabla v_h|)^{\frac{p(2-p)}{2}} \quad \text{for a.e. } x \in \Omega. \tag{3.62}$$

and

$$|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla v_h)|^2 \le \frac{c_4}{c_5}|\nabla u - \nabla v_h|^p \quad \text{for a.e. } x \in \Omega. \tag{3.63}$$

to the $W^{1,p}$ *seminorm or the quantity* $|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla u_h)|$*.*

Remark 3.10. *All the results derived above remain valid if* $f \in W^{-1,p'}(\Omega)$*. In this case, all integrals of the form* $\int_\Omega f w dx$ *for* $f \in L^{p'}(\Omega)$ *and* $w \in W_0^{1,p}(\Omega)$ *above are replaced by the duality product* $\langle f, w\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)}$*. In this case, by using Poincaré's inequality* (2.6a) *the estimates* $\int_\Omega f w dx \le C_P\|f\|_{L^{p'}(\Omega)}\|w\|_{L^p(\Omega)}$ *in the proofs above are just replaced by the estimate*

$$\langle f, w\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} \le \|f\|_{W^{-1,p'}(\Omega)}\|w\|_{W^{1,p}(\Omega)} \le (1+C_P^p)^{\frac{1}{p}}\|f\|_{W^{-1,p'}(\Omega)}\|\nabla w\|_{L^p(\Omega)}.$$

4 Convergence rates

4.1 Basic convergence rates under the assumption $u \in W^{l,p}(\Omega)$

So far in the derivation of the near-best approximation results (3.30) and (3.61) we have only assumed that u is the weak solution of (1.1), i.e., that $u \in W_g^{1,p}(\Omega)$. Let us further assume that $u \in W^{l,p}(\Omega)$, with $l \geq 2$. Then, setting $v_h = I_h^k u$ in (3.30) and (3.61) with $k \geq l-1$, and utilizing Lemma 3.1 we obtain the following (suboptimal) error estimates:

$$|u - u_h|_{W^{1,p}(\Omega)} \lesssim h^{\frac{(l-1)2}{p}} \quad \text{for } p > 2, \tag{4.1a}$$

$$|u - u_h|_{W^{1,p}(\Omega)} \lesssim h^{\frac{(l-1)p}{2}} \quad \text{for } 1 < p \leq 2, \tag{4.1b}$$

where $u_h \in V_{g,h}^{(k)}$ is the Galerkin approximation of u defined in (3.3).
Similarly, setting $v_h = I_h^k u$ in (3.30) (or in (3.40)) and in (3.61), and utilizing again Lemma 3.1, we obtain the bounds

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \lesssim h^{l-1} \quad \text{for } p > 2, \tag{4.2a}$$

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \lesssim h^{\frac{(l-1)p}{2}} \quad \text{for } 1 < p \leq 2, \tag{4.2b}$$

Remark 4.1. *Note that the discretization convergence rates* (4.1a)*,* (4.1b)*,* (4.2a)*,* (4.2b) *hold for any $\epsilon \geq 0$ in Assumption 2.*

4.2 Improved convergence rates for the case $1 < p \leq 2$

In this section we will only focus on the case $1 < p \leq 2$. From Section 4.1 it becomes clear that one cannot immediately obtain optimal convergence rates (see (4.1b) and (4.2b)) just by using the near-best approximation result (3.61), even if u was assumed to be in $W^{l,2}(\Omega)$. Here, by optimal rate it should be understood the convergence rate of the finite element approximation error in the corresponding norm. In the case of the $W^{1,p}$-seminorm, we know that the approximation error $\inf\limits_{v_h \in V_{g,h}} |u - v_h|_{W^{1,p}(\Omega)}$ has a convergence rate $\min(k+1, l) - 1$ if $u \in W^{l,p}(\Omega)$, but the rate that can be inferred from (3.61) (see (4.1b) for $k = l-1$) for the discretization error $|u - u_h|_{W^{1,p}(\Omega)}$ is only $(\min\{k+1, l\} - 1)p/2$. Likewise, the rate that can be directly inferred from (3.61) (see (4.2b) for $k = l-1$) for the discretization error $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$ is $(\min\{k+1, l\} - 1)p/2$. However, in what follows, we will see that the convergence rate of those discretization errors can be improved if $u \in W^{l,\bar{p}}(\Omega)$ with $\bar{p} \geq p$. In particular, we will see that if $\bar{p} \geq 2$, then the approximation error $\inf\limits_{v_h \in V_{g,h}} \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)}$ converges with a rate $\min(k+1, l) - 1$. This is done by modifying the derivation of the approximation error estimate (3.60) in Section 3.1.2 under the assumption $\epsilon > 0$. Consequently, thanks to the near-best approximation result (3.7), the discretization error in the $\mathbf{F}$-quasinorm converges with the same rate.

Assumption 4 (Additional assumption on the p-structure of φ)**.** The inequality (2.10) in Assumption 2 holds with $\epsilon > 0$.

4.2.1 Convergence rates for $u \in W^{l,\bar{p}}(\Omega)$ with $\bar{p} \geq p$ and $l > d/\bar{p}$

We make the following additional assumption on the regularity of the weak solution u of (2.24):

$$u \in W^{l,\bar{p}}(\Omega) \quad \text{for some } \bar{p} \geq p \text{ and } l \geq 2. \tag{4.3}$$

Let Assumption 4 hold and let u satisfy (4.3). We distinguish two cases. First, let $2 \geq \bar{p} \geq p$. Then, by using (2.21) and (2.10) we obtain

$$\begin{aligned}
\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|^2_{L^2(\Omega)} &\leq \frac{1}{c_5}\int_\Omega |\nabla u - \nabla v_h|^2 \varphi''\left(|\nabla u| + |\nabla v_h|\right) dx \\
&= \frac{1}{c_5}\int_\Omega |\nabla u - \nabla v_h|^{\bar{p}} \varphi''\left(|\nabla u| + |\nabla v_h|\right) |\nabla u - \nabla v_h|^{2-\bar{p}} dx \\
&\overset{(2.10)}{\leq} \frac{c_4}{c_5}\int_\Omega |\nabla u - \nabla v_h|^{\bar{p}} \frac{|\nabla u - \nabla v_h|^{2-\bar{p}}}{(\epsilon + |\nabla u| + |\nabla v_h|)^{2-p}} dx \\
&\leq \frac{c_4}{c_5}\int_\Omega |\nabla u - \nabla v_h|^{\bar{p}} \frac{(\epsilon + |\nabla u| + |\nabla v_h|)^{2-\bar{p}}}{(\epsilon + |\nabla u| + |\nabla v_h|)^{2-p}} dx \\
&\leq \frac{c_4}{c_5}\frac{1}{\epsilon^{\bar{p}-p}} |u - v_h|^{\bar{p}}_{W^{1,\bar{p}}(\Omega)} \qquad \text{for all } v_h \in V^{(k)}_{g,h},
\end{aligned} \tag{4.4}$$

and taking the square root we directly obtain

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|_{L^2(\Omega)} \leq \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}} \left(\frac{1}{\epsilon^{\bar{p}-p}}\right)^{\frac{1}{2}} |u - v_h|^{\frac{\bar{p}}{2}}_{W^{1,\bar{p}}(\Omega)} \quad \text{for all } v_h \in V^{(k)}_{g,h}. \tag{4.5}$$

Suppose that $l - d/\bar{p} > 0$. In this case $W^{l,\bar{p}}(\Omega)$ is embedded in $C^0(\overline{\Omega})$ and the interpolant of $I_h^k u$ is well defined (one can also make use of interpolants for low regularity solutions, e.g., Scott-Zhang). Setting $v_h = I_h^k u$ in (4.5), then using Lemma 3.1 and combining the result with the first two inequalities in (3.61) we obtain for $k \geq l - 1$

$$\begin{aligned}
|u - u_h|_{W^{1,p}(\Omega)} &\lesssim \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \\
&\lesssim \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_h^k u)\|_{L^2(\Omega)} \leq \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}} \left(\frac{1}{\epsilon^{\bar{p}-p}}\right)^{\frac{1}{2}} C^{\frac{\bar{p}}{2}}_{int,\bar{p},l,\bar{p}} h^{(l-1)\frac{\bar{p}}{2}} |u|^{\frac{\bar{p}}{2}}_{W^{l,\bar{p}}(\Omega)}.
\end{aligned} \tag{4.6}$$

On the other hand, when $\bar{p} > 2$ it is easy to see that

$$\begin{aligned}
\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|^2_{L^2(\Omega)} &\lesssim \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v_h)\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{c_5}\int_\Omega |\nabla u - \nabla v_h|^2 \varphi''\left(|\nabla u| + |\nabla v_h|\right) dx \\
&\leq \frac{c_4}{c_5}\frac{1}{\epsilon^{2-p}} |u - v_h|^2_{W^{1,2}(\Omega)} \quad \text{for all } v_h \in V^{(k)}_{g,h}.
\end{aligned} \tag{4.7}$$

Again, by taking $v_h = I_h^k u$ in (4.7), then using Lemma 3.1 and combining the result with the first two inequalities in (3.61) we obtain for $k \geq l-1$

$$\begin{aligned}|u-u_h|_{W^{1,p}(\Omega)} &\lesssim \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \\ &\lesssim \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla I_h^k u)\|_{L^2(\Omega)} \leq \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}}\left(\frac{1}{\epsilon^{2-p}}\right)^{\frac{1}{2}} C_{int,2,l,2}h^{l-1}|u|_{W^{l,2}(\Omega)}.\end{aligned} \tag{4.8}$$

4.2.2 Convergence rates for $u \in W^{l,\bar{p}}(\Omega)$ with $\bar{p} \geq 1$ and $l > 1 + d/\bar{p}$

Here we make the following additional assumption on the regularity of the weak solution u of (2.24) (for u being in $\in W_g^{1,p}(\Omega)$):

$$u \in W^{l,\bar{p}}(\Omega) \quad \text{for some } \bar{p} \geq 1 \text{ with } l \geq 2 \text{ and } l > 1 + d/\bar{p}. \tag{4.9}$$

In this case, since $l-1-d/\bar{p} > 0$ from the Sobolev embedding theorem (see Theorem 4.12 in [1]) we have that $u \in W^{1,\infty}(\Omega)$. This means that the analysis presented below also covers cases where $\bar{p} < p$. For example, if $d = 2$ and $l \geq 3$ we can have any $\bar{p} > 1$ and any $1 < p \leq 2$. Next, if $d = 3$ and $l = 3$ we can have any p and $\bar{p}$ such that $1.5 < \bar{p} < p$ in addition to the cases where $\bar{p} \geq p$ from Section 4.2.1 (see Example 7 in Section 5). If $d = 3$ and $l \geq 4$, then our analysis covers all $\bar{p} > 1$ and $1 < p \leq 2$ as well.

Let Assumption 4 hold and let u satisfy (4.9). Again, we distinguish two cases. First, let $2 \geq \bar{p} \geq 1$. Then, by using (2.10) we have

$$\begin{aligned}\|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla v_h)\|^2_{L^2(\Omega)} &\leq \frac{1}{c_5}\int_\Omega |\nabla u - \nabla v_h|^2 \varphi''(|\nabla u| + |\nabla v_h|)\,dx \\ &= \frac{1}{c_5}\int_\Omega |\nabla u - \nabla v_h|^{\bar{p}} \varphi''(|\nabla u| + |\nabla v_h|)|\nabla u - \nabla v_h|^{2-\bar{p}}\,dx \\ &\leq \frac{c_4}{c_5}\int_\Omega |\nabla u - \nabla v_h|^{\bar{p}} \frac{|\nabla u - \nabla v_h|^{2-\bar{p}}}{(\epsilon + |\nabla u| + |\nabla v_h|)^{2-p}}dx \\ &\leq \frac{c_4}{c_5}\frac{1}{\epsilon^{2-p}}|u-v_h|^{\bar{p}}_{W^{1,\bar{p}}(\Omega)}|u-v_h|^{2-\bar{p}}_{W^{1,\infty}(\Omega)} \quad \text{for all } v_h \in V_{g,h}^{(k)}.\end{aligned} \tag{4.10}$$

By setting $v_h = I_h^k u$ in (4.10), then using Lemma 3.1 and combining the result with the first two inequalities in (3.61) we obtain

$$\begin{aligned}&|u-u_h|_{W^{1,p}(\Omega)} \lesssim \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \lesssim \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla I_h^k u)\|_{L^2(\Omega)} \\ &\leq \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}}\left(\frac{1}{\epsilon^{2-p}}\right)^{\frac{1}{2}} C^{\frac{\bar{p}}{2}}_{int,\bar{p},l,\bar{p}} h^{\frac{(l-1)\bar{p}}{2}}|u|^{\frac{\bar{p}}{2}}_{W^{l,\bar{p}}(\Omega)} C^{\frac{2-\bar{p}}{2}}_{int,\infty,l,\bar{p}} h^{\frac{(l-1-d/\bar{p})(2-\bar{p})}{2}}|u|^{\frac{2-\bar{p}}{2}}_{W^{l,\bar{p}}(\Omega)} \\ &= \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}}\left(\frac{1}{\epsilon^{2-p}}\right)^{\frac{1}{2}} C^{\frac{\bar{p}}{2}}_{int,\bar{p},l,\bar{p}} C^{\frac{2-\bar{p}}{2}}_{int,\infty,l,\bar{p}} h^{l-1+\frac{d}{2}-\frac{d}{\bar{p}}}|u|_{W^{l,\bar{p}}(\Omega)}.\end{aligned} \tag{4.11}$$

If $\bar{p} \geq 2$, it is easy to see that

$$\begin{aligned}|u-u_h|_{W^{1,p}(\Omega)} &\lesssim \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \\ &\lesssim \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla I_h^k u)\|_{L^2(\Omega)} \\ &\leq \left(\frac{c_4}{c_5}\right)^{\frac{1}{2}}\left(\frac{1}{\epsilon^{2-p}}\right)^{\frac{1}{2}} C_{int,2,l,2}h^{l-1}|u|_{W^{l,2}(\Omega)}.\end{aligned} \tag{4.12}$$

We can summarize the expected convergence rates from Section 4.2.1 and Section 4.2.2 in the following theorem.

Theorem 4.2. *Let φ satisfy Assumption 1 and Assumption 2 for some $1 < p \leq 2$. Let us further suppose that Assumption 4 holds. Let $u \in W_g^{1,p}(\Omega)$ be the solution of* (2.24) *and let $u_h \in V_{g,h}^{(k)}$ be the Galerkin approximation of u defined by* (3.3)*.*

- *If $u \in W^{l,\bar{p}}(\Omega)$ with $\bar{p} \geq p$ and $l > d/\bar{p}$, then for $k \geq l-1$*

$$\begin{aligned} |u - u_h|_{W^{1,p}(\Omega)} &\lesssim \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \\ &\lesssim \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_h^k u)\|_{L^2(\Omega)} \leq C_2(\epsilon) h^{\min\left\{(l-1)\frac{\bar{p}}{2}, l-1\right\}}. \end{aligned} \tag{4.13}$$

- *If $u \in W^{l,\bar{p}}(\Omega)$ with $\bar{p} \geq 1$ and $l > 1 + d/\bar{p}$, then for $k \geq l-1$*

$$\begin{aligned} |u - u_h|_{W^{1,p}(\Omega)} &\lesssim \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)} \\ &\lesssim \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_h^k u)\|_{L^2(\Omega)} \leq C_3(\epsilon) h^{l-1+\min\left\{\frac{d}{2}-\frac{d}{\bar{p}}, 0\right\}}. \end{aligned} \tag{4.14}$$

The constants $C_2(\epsilon)$ and $C_3(\epsilon)$ diverge as $\epsilon \to 0$. More explicit expressions for these constants can be found by comparing (4.13) *to* (4.6) *and* (4.8)*, and* (4.14) *to* (4.11) *and* (4.12)*.*

Remark 4.3. • *Note that the estimate* (4.13) *agrees with the estimate* (3.9)*. Indeed, from Theorem 4.2 it follows that if $u \in W^{2,2}(\Omega)$, then the convergence rate in* (4.13) *is the optimal $O(h^1)$ for $\mathbb{P}_1$ finite elements. On the other hand, from Remark 2.5 it follows that $\mathbf{F}(\nabla u) \in W^{1,2}(\Omega)$ and thus* (3.9) *is also valid giving the optimal $O(h^1)$ convergence rate.*

• *Also, notice that the estimate* (3.9) *given in Proposition 3.2 is valid for any $1 < p < \infty$ and $\epsilon \geq 0$, whereas the estimates in Theorem 4.2 are derived under the assumption $1 < p \leq 2$ and $\epsilon > 0$. However, at the cost of the additional assumption on ϵ, Theorem 4.2 also gives the convergence rates for the case of higher regularity on u and higher order polynomial spaces. In particular, it gives optimal convergence rates when $u \in W^{l,2}(\Omega)$ with $l \geq 2$ and polynomial spaces $\mathbb{P}_k$ with $k \geq l-1$.*

Remark 4.4 (Optimal rates in the case $p > 2$)**.** *We should also emphasize that according to the estimate* (4.2a) *in the case where $p > 2$, one immediately obtains the optimal $O(h^1)$ convergence rate if $u \in W^{2,p}(\Omega)$. The estimate* (4.2a) *also gives optimal convergence rates $O(h^{l-1})$ in case of higher regularity on u, i.e., $u \in W^{l,p}(\Omega)$ and higher order polynomial spaces. Moreover, this estimate is valid for any $p > 2$ and any $\epsilon \geq 0$ in Assumption 1.*

On the other hand, Assumption 2 combined with the equivalences (2.20) *imply that $\mathbf{F}$ is locally Lipschitz:*

$$\begin{aligned} |\mathbf{F}(\mathbf{a}) - \mathbf{F}(\mathbf{b})| &\lesssim \sqrt{\varphi''(|\mathbf{a}| + |\mathbf{b}|)}\, |\mathbf{a} - \mathbf{b}| \\ &\leq \left(c_4 \left(\epsilon + |\mathbf{a}| + |\mathbf{b}|\right)^{p-2}\right)^{\frac{1}{2}} |\mathbf{a} - \mathbf{b}| \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^d. \end{aligned} \tag{4.15}$$

Now, if $d \in \{2, 3\}$ and $u \in W^{2,p}(\Omega)$ with $p > d$ (i.e., $\nabla u \in [W^{1,p}(\Omega)]^d$), from Theorem 1 in [25] applied to each component of the vector function $\mathbf{F} = (F_1, \ldots, F_d)$ it follows that $\mathbf{F}(\nabla u) \in [W^{1,p}(\Omega)]^d \subset [W^{1,2}(\Omega)]^d$. This means that for $p > d$ our result again agrees with the estimate (3.9) *derived in [21].*

5 Numerical examples

In this section, we present a series of numerical examples which confirm the predicted convergence rates, given in Theorem 4.2. More precisely, we consider (1.1) for φ' and $\mathbf{A}$ as in Example 1 in Section 2.2 with a prescribed solution $u = |x|^{\gamma}$, $x = (x_1, \dots, x_d)$ for some $\gamma > 0$. In this case, the regularity of the solution depends only on the parameter γ. It can be checked that $u \in W^{l,\bar{p}}(\Omega)$ if and only if $\gamma > l - \frac{d}{\bar{p}}$. For a given parameter γ and a differentiability exponent l we denote by $\bar{p}$ the limit summability of the solution u, given by $\bar{p} = d/(l-\gamma)$.

The problem is posed on the domain $\Omega = (-L, L)^d$ with $L = 0.5$ for $d = 2$ and $L = 0.4$ for $d = 3$. Here, we note that for the above defined solution u the boundary data is not in the local FE space on the boundary $\partial\Omega$ and thus Assumption 3 is not satisfied. However, this does not affect the convergence rates obtained below. In fact, to be precise and satsify Assumption 3 one can consider the function $\overline{u}(x) = \psi(x)u(x) = \psi(x)\,|x|^{\gamma}$ instead of $u = |x|^{\gamma}$, where ψ is a smooth function[2] in Ω with $\psi(x) = 0$ on $\partial\Omega$ such that ψ is either uniformly positive or uniformly negative in a neighborhood of $0 \in \mathbb{R}^d$. The problem is solved using local polynomial spaces $\mathbb{P}_k$ with $k = 1$, $k = 2$ and $k = 3$. The nonlinear algebraic system resulting from (3.3) is linearized by applying a simple Picard iterative method and the final derived linear system is solved by a Jacobi preconditioned conjugate gradient method with a very high accuracy. Every example has been solved by applying several refinement steps on structured meshes with $h_s, h_{s+1}, \dots$, starting from $h_{s=0} = a\sqrt{d}$ with $a = 0.1$ for $d = 2$ and $a = 0.4$ for $d = 3$. The numerical convergence rates r_D^F are computed by the formula

$$r_D^F = d\frac{\ln\left(e_{s-1}/e_s\right)}{\ln\left(n_s/n_{s-1}\right)}, \text{ with } e_s = \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}, \tag{5.1a}$$

and compared with the corresponding interpolation rates r_I^F given by

$$r_I^F = d\frac{\ln\left(e_{s-1}/e_s\right)}{\ln\left(n_s/n_{s-1}\right)}, \text{ with } e_s = \|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}, \tag{5.1b}$$

where n_s is the number of nodes at mesh refinement level (MRL) s. The rates are given in the respective tables below. When the calculations are not feasible because of computational reasons related to memory, the corresponding entries in the tables are left empty.

Remark 5.1. *Notice that if $\bar{p}$ is the limit summability exponent of u for some fixed l and γ, then u does not really belong to the space $W^{l,\bar{p}}(\Omega)$, but rather to the space $W^{l,\bar{p}-\delta}(\Omega)$ for any $0 < \delta \le \bar{p} - 1$. Nevertheless, the predicted convergence rates in* (4.13) *and* (4.14) *should be computed with the limit value $\bar{p}$. In order to be precise, we will write $u \in W^{l,\bar{p}-\delta}(\Omega)$ for all δ such that $0 < \delta \le \bar{p} - 1$ instead of just $u \in W^{l,\bar{p}}(\Omega)$ in the tables below.*

Because we are going to study the behavior of the convergence rates and check the validity of the theoretical approximation estimates, we add below a proposition which introduces a condition on γ in order $\mathbf{F}(\nabla u) \in \left[W^{1,q}(\Omega)\right]^d$, $q > 1$ for $u = |x|^{\gamma}$.

[2] An example of such a function for the domain $\Omega = (-A, A)^d$ is for example $\psi(x) = \cos(\frac{\pi}{2A}x_1)\dots\cos(\frac{\pi}{2A}x_d)$.

Proposition 5.2. *Let $u = |x|^\gamma$, $\epsilon \geq 0$ and $1 < p \leq 2$. If $\mathbf{F}$ is given by (2.18), then for any fixed $q > 1$ it holds $\mathbf{F}(\nabla u) \in [W^{1,q}(\Omega)]^d$ for all γ satisfying*

$$\gamma > \frac{pq + 2q - 2d}{pq}. \tag{5.2}$$

Proof. We first compute the weak gradient of u by using the absolute continuity (AC) characterization on lines for Sobolev functions (see Theorem 10.35 in [22]):

$$\nabla u = \gamma |x|^{\gamma-1} \frac{x}{|x|}. \tag{5.3}$$

We have $\mathbf{F}(\nabla u) := (F_1(\nabla u), \dots, F_d(\nabla u))$, where

$$F_i(\nabla u) = (\epsilon + |\nabla u|)^{\frac{p-2}{2}} \frac{\partial u}{\partial x_i} = \left(\epsilon + \gamma |x|^{\gamma-1}\right)^{\frac{p-2}{2}} \gamma |x|^{\gamma-2} x_i, \; i = 1, \dots, d. \tag{5.4}$$

Since $p \leq 2$, we can estimate

$$|F_i(\nabla u)| \lesssim |x|^{\gamma-1+(\gamma-1)\left(\frac{p-2}{2}\right)}. \tag{5.5}$$

By using spherical coordinates, it is easy to see[3] that if

$$\gamma - 1 + (\gamma - 1)\left(\frac{p-2}{2}\right) > -\frac{d}{q}, \tag{5.6}$$

then $F_i(\nabla u) \in L^q(\Omega)$. The last condition is equivalent to $\gamma > 1 - (2d)/(pq)$. Now, we proceed with the weak derivatives of $F_i(\nabla u)$, $i = 1, \dots d$, for which we also use the AC characterization of Sobolev functions:

$$\begin{aligned} \frac{\partial F_i(\nabla u)}{\partial x_j} &= \left(\frac{p-2}{2}\right) \left(\epsilon + \gamma |x|^{\gamma-1}\right)^{\frac{p-4}{2}} \gamma^2 (\gamma - 1) |x|^{2\gamma-5} x_i x_j \\ &\quad + \left(\epsilon + \gamma |x|^{\gamma-1}\right)^{\frac{p-2}{2}} \gamma(\gamma - 2) |x|^{\gamma-4} x_i x_j \\ &\quad + \left(\epsilon + \gamma |x|^{\gamma-1}\right)^{\frac{p-2}{2}} \gamma |x|^{\gamma-2} \delta_{i,j} =: I_1 + I_2 + I_3. \end{aligned} \tag{5.7}$$

From (5.7) it is clear that $\frac{\partial F_i(\nabla u)}{\partial x_j}$ will be in $L^q(\Omega)$ as long as I_1, I_2, I_3 are in $L^q(\Omega)$. We have

$$|I_1| \lesssim |x|^{2\gamma-3+(\gamma-1)\left(\frac{p-4}{2}\right)}, \; |I_2| \lesssim |x|^{\gamma-2+(\gamma-1)\left(\frac{p-2}{2}\right)}, \; |I_3| \lesssim |x|^{\gamma-2+(\gamma-1)\left(\frac{p-2}{2}\right)}. \tag{5.8}$$

Finally, I_1, I_2, and I_3 will be in $L^q(\Omega)$ as long as the following conditions are satisfied

$$2\gamma - 3 + (\gamma - 1)\left(\frac{p-4}{2}\right) > -\frac{d}{q}, \qquad \gamma - 2 + (\gamma - 1)\left(\frac{p-2}{2}\right) > -\frac{d}{q}. \tag{5.9}$$

From (5.9) we obtain

$$\gamma > \frac{pq + 2q - 2d}{pq}. \tag{5.10}$$

□

[3] For any d, using spherical coordinates, we have $\int\limits_{B(0,1)} |x|^\alpha \, dx \sim \int\limits_0^1 \rho^\alpha \rho^{d-1} d\rho = \int\limits_0^1 \rho^{\alpha+d-1} d\rho < \infty$ if and only if $\alpha + d - 1 > -1$, i.e., if and only if $\alpha > -d$.

Example 1. $\mathbf{d = 2}$, $\mathbf{u \in W^{l,\bar{p}-\delta}(\Omega)}$, $\mathbf{l = 2}$, $\mathbf{p = 1.2}$, ***and*** $\mathbf{p < \bar{p} < 2}$.

The first numerical example is a simple test case validating the theoretical convergence rates for $l = 2$. Here, the parameter p in Assumption 2 is equal to 1.2. We perform two groups of computations corresponding to $\gamma = 0.35$ and $\gamma = 0.7$, where for the first case we get $\bar{p} \approx 1.212$ and for the second $\bar{p} \approx 1.538$. The expected convergence rates in this example are given by the estimate (4.13). The values of the rest parameters are given in the first lines in Table 1. The numerical convergence rates r_D^F and r_I^F for several levels of mesh refinement are given in Table 1. They are in a very good agreement and also agree with the theoretically predicted estimates given in Theorem 4.2.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \lvert x\rvert^\gamma$, $\epsilon = 1$, $p = 1.2$, $d = 2$, $h_0 = 0.1\sqrt{2}$												
h_0	$\gamma = 0.35$ $l = 2$ $\bar{p} \approx 1.212$						$\gamma = 0.7$ $l = 2$ $\bar{p} \approx 1.538$					
$h_s = \frac{h_0}{2^s}$	$k = 1$		$k = 2$		$k = 3$		$k = 1$		$k = 2$		$k = 3$	
	Expected rates r_D^F and r_I^F											
	$r_D^F = r_I^F \approx 0.606$						$r_D^F = r_I^F \approx 0.769$					
	Computed rates r_D^F and r_I^F											
	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-	-	-	-	-	-	-	-	-
$s = 1$	0.554	0.618	0.606	0.597	0.621	0.632	0.792	0.792	0.847	0.847	0.844	0.853
$s = 2$	0.563	0.609	0.598	0.590	0.610	0.619	0.782	0.782	0.823	0.823	0.821	0.829
$s = 3$	0.572	0.606	0.597	0.591	0.606	0.614	0.781	0.781	0.813	0.813	0.810	0.818
$s = 4$	0.580	0.605	0.599	0.594	0.606	0.611	0.783	0.783	0.809	0.810	0.806	0.814
$s = 5$	0.586	0.605	0.601	0.598	0.606	0.610	0.787	0.787	0.809	0.809	0.805	0.812
$s = 6$	0.592	0.606	0.603	0.601	0.607	0.610	0.791	0.791	0.809	0.810	0.806	0.813
$s = 7$	0.596	0.607	0.605	0.604	0.608	0.609	0.795	0.795	0.810	0.811	0.807	0.811
$s = 8$	0.600	0.607	-	-	-	-	0.798	0.798	-	-	-	-
$s = 9$	0.603	0.608	-	-	-	-	-	-	-	-	-	-

Table 1: Example 5: Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Example 2. $\mathbf{d = 2}$, $\mathbf{u \in W^{l,\bar{p}-\delta}(\Omega)}$, $\mathbf{l \in \{3, 4\}}$, $\mathbf{p = 1.2}$, ***and*** $\mathbf{1 < \bar{p} < 2}$.

In the second example, we consider problem (1.1) with $p = 1.2$ in Assumption 2. As a first step, we are interested in examining the behavior of the convergence rates for prescribed solutions u with higher differentiability and a limit summability $\bar{p} \geq p$. We perform two groups of computations which correspond to two different values $l = 3$ and $l = 4$ for the differentiability. Thus, for the first group we chose $\gamma \in \{1.35, 1.7\}$ which gives $\bar{p} \approx 1.212$ and $\bar{p} \approx 1.538$, respectively, and for the second group we set $\gamma \in \{2.35, 2.7\}$ with $\bar{p} \approx 1.212$ and $\bar{p} \approx 1.538$, respectively. The numerical convergence rates are computed using the estimate given in (4.14).

Since the regularity of the solution is high we perform computations using $k = 2$ and $k = 3$, satisfying $k \geq l - 1$. The values of the rest parameters and the expected convergence rates are given in Table 2. The computed convergence rates and the related interpolation rates are shown in Table 2. We note that despite the use of high order polynomial spaces the convergence is suboptimal due to the regularity of the solution. In any case, we observe that the numerical convergence rates take values very close to the predicted rates for all mesh refinement levels, and this validates the theoretical results.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \|x\|^\gamma$, $\epsilon = 1$, $p = 1.2$, $d = 2$, $h_0 = 0.1\sqrt{2}$												
h_0	$\gamma = 1.35$ $l = 3$ $\bar{p} \approx 1.212$				$\gamma = 1.7$ $l = 3$ $\bar{p} \approx 1.538$				$\gamma = 2.35$ $l = 4$ $\bar{p} \approx 1.212$		$\gamma = 2.7$ $l = 4$ $\bar{p} \approx 1.538$	
$h_s = \frac{h_0}{2^s}$	$k = 2$		$k = 3$		$k = 2$		$k = 3$		$k = 3$		$k = 3$	
	Expected rates r_D^F and r_I^F											
	$r_D^F = r_I^F = 1.35$				$r_D^F = r_I^F = 1.7$				$r_D^F = r_I^F = 2.35$		$r_D^F = r_I^F = 2.7$	
	Computed rates r_D^F and r_I^F											
	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-	-	-	-	-	-	-	-	-
$s = 1$	1.381	1.385	1.396	1.393	1.691	1.695	1.752	1.766	2.460	2.458	2.797	2.798
$s = 2$	1.346	1.349	1.357	1.354	1.671	1.672	1.715	1.724	2.406	2.406	2.740	2.741
$s = 3$	1.332	1.334	1.340	1.337	1.666	1.668	1.701	1.706	2.379	2.379	2.714	2.714
$s = 4$	1.327	1.329	1.334	1.331	1.669	1.670	1.696	1.699	2.365	2.365	2.702	2.703
$s = 5$	1.327	1.328	1.333	1.331	1.675	1.675	1.695	1.697	2.357	2.357	2.698	2.698
$s = 6$	1.329	1.330	1.334	1.332	1.680	1.681	1.696	1.697	2.354	2.354	2.697	2.697
$s = 7$	1.332	1.333	1.336	1.335	1.685	1.685	-	-	-	-	-	-
$s = 8$	1.335	1.335	-	-	1.689	1.689	-	-	-	-	-	-
$s = 9$	1.337	1.338	-	-	-	-	-	-	-	-	-	-

Table 2: Example 2: Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Next, we present computations for the case where $\bar{p} < p$ and $l > 1 + d/\bar{p}$ with $p = 1.2$ for all computations. For these test cases, the approximation estimates are given by (4.14). We show two tests where the values of the related parameters are $\{\gamma = 1.2,\, l = 3,\, \bar{p} \approx 1.111\}$ and $\{\gamma = 2.2,\, l = 4,\, \bar{p} \approx 1.111\}$ respectively. For both test cases the limit summability is $\bar{p} \approx 1.111 < p$. The computations have been performed using local polynomial degree $k = l - 1$ of the finite element spaces. The numerical convergence rates are presented in Table 3. We observe that the rates are determined by the reduced regularity of u and are suboptimal with respect to the polynomial degree k. For both computations the discretization error rates are in a very good agreement with the associated interpolation rates and are in agreement with the theoretically predicted rates.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \|x\|^{\gamma}$, $\epsilon = 1$, $p = 1.2$, $d = 2$, $h_0 = 0.1\sqrt{2}$				
h_0	$\gamma = 1.2$ $l = 3$ $\bar{p} \approx 1.111$		$\gamma = 2.2$ $l = 4$ $\bar{p} \approx 1.111$	
$h_s = \frac{h_0}{2^s}$	$k = 2$		$k = 3$	
	Expected rates r_D^F and r_I^F			
	$r_D^F = r_I^F = 1.2$		$r_D^F = r_I^F = 2.2$	
	Computed rates r_D^F and r_I^F			
	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-
$s = 1$	1.238	1.239	2.303	2.301
$s = 2$	1.204	1.204	2.252	2.251
$s = 3$	1.188	1.188	2.227	2.226
$s = 4$	1.180	1.180	2.213	2.213
$s = 5$	1.178	1.178	2.207	2.207
$s = 6$	1.178	1.177	2.203	2.203
$s = 7$	1.179	1.178	2.201	2.201
$s = 8$	1.180	1.180	2.200	2.200

Table 3: Example 2: Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Example 3. $\mathbf{d = 2}$, $\mathbf{u \in W^{l,\bar{p}-\delta}(\Omega)}$, $\mathbf{l \in \{2, 3, 4\}}$ *and* $\mathbf{p = 1.5}$, $\mathbf{1 < \bar{p} < 2}$.

In this example, we investigate the behavior of the convergence rates for high differentiability of u and $\bar{p} > p$. We have performed three groups of computations for $l \in \{2, 3, 4\}$ setting $\gamma \in \{0.7, 1.7, 2.7\}$ respectively. The corresponding value of $\bar{p}$ for all tests is $\bar{p} \approx 1.538$. The discretization error is computed for $p = 1.5$ in Assumption 2. The convergence rates are displayed in Table 4. We can observe that the convergence rates r_D^F are in a very good agreement with the corresponding rates r_I^F, and coincide with the expected convergence rates. This test also verifies the theoretically predicted estimates in Theorem 4.2. It is worth pointing out that the convergence rates r_D^F and r_I^F do not depend on the parameter p. This can be seen from Theorem 4.2 and also can be observed numerically by comparing the results for the second and third groups of computations to the computations corresponding to $\gamma = 1.7$ and $\gamma = 2.7$ in Example 2.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \lvert x\rvert^{\gamma}$, $\epsilon = 1$, $p = 1.5$, $d = 2$, $h_0 = 0.1\sqrt{2}$												
h_0	$\gamma = 0.7$ $l = 2$ $\bar{p} \approx 1.538$						$\gamma = 1.7$ $l = 3$ $\bar{p} \approx 1.538$				$\gamma = 2.7$ $l = 4$ $\bar{p} \approx 1.538$	
$h_s = \frac{h_0}{2^s}$	$k = 1$		$k = 2$		$k = 3$		$k = 2$		$k = 3$		$k = 3$	
	Expected rates r_D^F and r_I^F											
	$r_D^F = r_I^F \approx 0.769$						$r_D^F = r_I^F = 1.7$				$r_D^F = r_I^F = 2.7$	
	Computed rates r_D^F and r_I^F											
	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-	-	-	-	-	-	-	-	-
$s = 1$	0.717	0.751	0.808	0.811	0.810	0.815	1.734	1.738	1.785	1.783	2.813	2.811
$s = 2$	0.725	0.746	0.785	0.788	0.785	0.790	1.700	1.703	1.736	1.735	2.747	2.748
$s = 3$	0.732	0.747	0.774	0.776	0.774	0.778	1.688	1.689	1.714	1.714	2.718	2.718
$s = 4$	0.739	0.750	0.769	0.772	0.770	0.773	1.685	1.686	1.705	1.704	2.705	2.705
$s = 5$	0.745	0.753	0.768	0.770	0.768	0.771	1.686	1.687	1.701	1.701	2.699	2.700
$s = 6$	0.750	0.757	0.768	0.770	0.768	0.771	1.688	1.689	1.699	1.699	2.696	2.698
$s = 7$	0.755	0.760	0.769	0.770	0.769	0.771	1.691	1.691	1.699	1.699	2.696	2.697
$s = 8$	0.758	0.762	-	-	-	-	1.693	1.693	-	-	-	-
$s = 9$	0.761	0.765	-	-	-	-	1.695	1.695	-	-	-	-

Table 4: Example 3 : Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Example 4. $\mathbf{d = 2}$, $\mathbf{u \in W^{l,\bar{p}-\delta}(\Omega)}$, $\mathbf{l \in \{2, 3, 4\}}$, $\mathbf{\bar{p} \geq 2}$.

In this collection of numerical results we set $\gamma \in \{1.2, 2.2, 3.2\}$ with corresponding values for the differentiability parameter $l \in \{2, 3, 4\}$, and with a limit summability $\bar{p} = 2.5$ for all test cases. The problem (1.1) is solved with the parameter p taking values 1.2 and 1.5 in Assumption 2. For each value of the differentiability parameter $l \in \{2, 3, 4\}$ we solve the problem using polynomial spaces with $k = l - 1$. The results are presented in Table 5. The convergence rates of the discretization error r_D^F are almost identical with the corresponding interpolation rates r_I^F, and very close to the expected rates given by (4.13). We point out that for all groups of computations, the convergence rates are optimal with respect to the polynomial order. This is expected since for each case we have $k = l - 1$ and $\bar{p} \geq 2$ (see (4.13)).

It is worth noting that the test corresponding to the parameter $\gamma = 1.2$ is performed with the same solution $u = |x|^{\gamma}$ as the one used in the first test in Table 3. The difference is that in Table 3 the solution has been chosen such that $u \in W^{l_1,\bar{p}_1}(\Omega)$ with differentiability $l_1 = 3$ and a limit summability $\bar{p}_1 \approx 1.111$, whereas the same solution u here is considered with the regularity $W^{l_2,\bar{p}_2}(\Omega)$ with differentiability $l_2 = 2$ and a limit summability $\bar{p}_2 = 2.5$. Notice that in both cases we have $\gamma = 1.2 = l_1 - d/\bar{p}_1 = l_2 - d/\bar{p}_2$. Similarly, the solution $u = |x|^{\gamma=2.2}$ lies in both spaces[4] $W^{l_1,\bar{p}_1}(\Omega)$ with $l_1 = 4$, $\bar{p}_1 \approx 1.111$ and in $W^{l_2,\bar{p}_2}(\Omega)$ with $l_2 = 3$, $\bar{p}_2 = 2.5$. By comparing the convergence rates in Table 3 and Table 5 (or just comparing the rates predicted by Theorem 4.2) for $\gamma \in \{1.2, 2.2\}$ and $p = 1.2$, it can be seen that the increase of the polynomial degree from $k \in \{1, 2\}$ to $k \in \{2, 3\}$ offers only 0.2 increase to the convergence rates. Therefore, for a given regularity of the solution u

[4]This can also be seen by using the Sobolev embedding theorem.

one can decide based on Theorem 4.2 which is the appropriate polynomial degree k in order to get the highest accuracy per degree of freedom.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \lvert x\rvert^\gamma$, $\epsilon = 1$, $\bar{p} = 2.5$, $d = 2$, $h_0 = 0.1\sqrt{2}$												
h_0	$\gamma = 1.2$, $l = 2$				$\gamma = 2.2$, $l = 3$				$\gamma = 3.2$, $l = 4$			
	$p = 1.2$		$p = 1.5$		$p = 1.2$		$p = 1.5$		$p = 1.2$		$p = 1.5$	
$\frac{h_0}{2^s}$	$k = 1$		$k = 1$		$k = 2$		$k = 2$		$k = 3$		$k = 3$	
	Expected rates r_D^F and r_I^F											
	$r_D^F = r_I^F = 1$				$r_D^F = r_I^F = 2$				$r_D^F = r_I^F = 3$			
	Computed rates r_D^F and r_I^F											
	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-	-	-	-	-	-	-	-	-
$s = 1$	0.963	0.971	0.967	0.977	2.051	2.056	2.066	2.068	3.119	3.121	3.126	3.126
$s = 2$	0.961	0.966	0.965	0.971	2.010	2.013	2.019	2.021	3.045	3.046	3.048	3.049
$s = 3$	0.964	0.968	0.968	0.972	1.993	1.995	1.999	2.000	3.010	3.011	3.012	3.013
$s = 4$	0.969	0.972	0.972	0.975	1.988	1.988	1.991	1.992	2.995	2.996	2.997	2.997
$s = 5$	0.975	0.977	0.977	0.979	1.987	1.987	1.989	1.990	2.990	2.991	2.991	2.992
$s = 6$	0.980	0.981	0.982	0.983	1.988	1.988	1.990	1.990	2.990	2.990	2.990	2.991
$s = 7$	0.984	0.985	0.985	0.987	1.990	1.990	1.991	1.991	-	-	-	-
$s = 8$	0.987	0.988	0.989	0.990	1.991	1.992	1.992	1.993	-	-	-	-

Table 5: Example 4: Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Example 5. $\mathbf{d = 2}$, $\epsilon = \mathbf{0.1}$, $\mathbf{p < \bar{p} < 2}$

In this test we check the asymptotic behavior of the rates for small ϵ. Since the constants $C_2(\epsilon)$ and $C_3(\epsilon)$ in the estimates of Theorem 4.2 blow up as ϵ goes to zero, we can expect that for small $\epsilon > 0$ more mesh refinement levels are necessary in order to observe the predicted (asymptotic) convergence rates. One can also expect that for $\epsilon = 0$ the convergence rates will be strictly smaller than the corresponding ones for positive ϵ.

We perform two groups of tests for $p = 1.2$. In the first group the parameters are $\{\gamma = 0.7, l = 2, \bar{p} \approx 1.538\}$ and for the second group $\{\gamma = 1.7, l = 3, \bar{p} \approx 1.538\}$. In Table 6 the numerical convergence rates are compared with the associated interpolation rates. We can see a good agreement between the interpolation rates and the rates of the numerical solution. In general the computed rates in Table 6 agree with the theoreticallly predicted rates for $\epsilon > 0$, see Theorem 4.2, and also agree with the rates obtained in Example 5 and Example 2 for the same values of the parameters.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \lvert x\rvert^{\gamma}$, $\epsilon = 0.1$, $p = 1.2$, $d = 2$, $h_0 = 0.1\sqrt{2}$						
h_0	$\gamma = 0.7$ $l = 2$ $\bar{p} \approx 1.538$		$\gamma = 1.7$ $l = 3$ $\bar{p} \approx 1.538$			
$h_s = \frac{h_0}{2^s}$	$k = 1$		$k = 2$		$k = 3$	
	Expected rates r_D^F and r_I^F					
	$r_D^F = r_I^F \approx 0.769$		$r_D^F = r_I^F = 1.7$			
	Computed rates r_D^F and r_I^F					
	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-	-	-
$s = 1$	0.758	0.791	3.835	1.581	5.673	1.619
$s = 2$	0.771	0.788	1.563	1.563	1.715	1.595
$s = 3$	0.779	0.790	1.559	1.566	1.595	1.598
$s = 4$	0.786	0.794	1.575	1.580	1.613	1.614
$s = 5$	0.792	0.798	1.596	1.599	1.633	1.634
$s = 6$	0.798	0.802	1.617	1.618	1.652	1.652
$s = 7$	0.802	0.805	1.635	1.636	1.667	1.667
$s = 8$	0.806	0.808	1.651	1.652	1.678	1.678
$s = 9$	0.808	0.811	-	-	-	-

Table 6: Example 5: $\epsilon = 0.1$: Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Example 6. $\mathbf{d = 2}$, $\epsilon = \mathbf{0}$, $\mathbf{p < \bar{p} \leq 2.5}$

Here, we present a collection of several tests with $\epsilon = 0$ in Assumption 2. In this case, we expect that the convergence rates will be no higher than those obtained for positive ϵ in Theorem 4.2 because the constants $C_2(\epsilon)$ and $C_3(\epsilon)$ diverge as $\epsilon \to 0^+$. We perform five two-dimensional tests where γ takes values in $\{0.7, 1.7, 2.7, 1.2, 2.2\}$ respectively, and we use $p = 1.2$ in Assumption 2. In some of the previous computational tests we have used the same values of γ but with $\epsilon = 1$ and so we can make a comparison of the results. The results for $\epsilon = 0$ are presented in Table 7.

Remark 5.3. *Note that for $\epsilon \geq 0$ in Assumption 2, it follows from Proposition 3.2, see (3.9), that if $\mathbf{F}(\nabla u) \in W^{2,2}(\Omega)$ then we also have optimal $O(h^1)$ convergence rates for the quantities $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \Pi_h^1 u)\|_{L^2(\Omega)}$ in case of $\mathbb{P}_1$ spaces. On the other hand, for $u = |x|^{\gamma}$ we have seen in Proposition 5.2 that if $\gamma > 1$ for $d = 2$ and $\gamma > 1 - 1/p$ for $d = 3$, then it holds $\mathbf{F}(\nabla u) \in [W^{1,2}(\Omega)]^d$ for any $\epsilon \geq 0$. Thus, according to Proposition 3.2 for such γ we expect convergence rates $r_D^F = r_I^F \geq 1$ for polynomial degree $k \geq 1$.*

Based on Remark 5.3, we can see in Table 7 that indeed for the tests with $k > 1$ the corresponding rates r_D^F, r_I^F are greater than one, but are strictly smaller than the corresponding rates obtained for the same tests for $\epsilon > 0$ (see, e.g., Table 2). This also indicates the importance of Assumption 4.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \lvert x\rvert^{\gamma}$, $\epsilon = 0$, $p = 1.2$, $d = 2$, $h_0 = 0.1\sqrt{2}$												
h_0	$\gamma = 0.7$ $l = 2$ $\bar{p} \approx 1.538$		$\gamma = 1.7$ $l = 3$ $\bar{p} \approx 1.538$				$\gamma = 2.7$ $l = 4$ $\bar{p} \approx 1.538$		$\gamma = 1.2$ $l = 2$ $\bar{p} = 2.5$		$\gamma = 2.2$ $l = 3$ $\bar{p} = 2.5$	
$h_s = \frac{h_0}{2^s}$	$k = 1$		$k = 2$		$k = 3$		$k = 3$		$k = 1$		$k = 2$	
	Expected rates r_D^F and r_I^F are between the basic convergence rates computed by (4.2b) and the improved convergence rates for $\epsilon > 0$ in Theorem 4.2											
	$r_D^F = r_I^F \in \left[\frac{p}{2}, 0.769\right]$		$p \leq r_D^F = r_I^F \leq 1.7$				$r_D^F = r_I^F \in \left[\frac{3}{2}p, 2.7\right]$		$r_D^F = r_I^F \in \left[\frac{p}{2}, 1\right]$		$r_D^F = r_I^F \in [p, 2]$	
	Computed rates r_D^F and r_I^F											
	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-	-	-	-	-	-	-	-	-
$s = 1$	0.813	0.813	1.684	1.513	4.871	1.521	6.153	2.162	1.056	0.951	5.293	1.804
$s = 2$	0.803	0.803	1.467	1.467	3.698	1.471	3.604	2.091	0.948	0.949	2.598	1.757
$s = 3$	0.801	0.801	1.443	1.443	1.661	1.445	2.082	2.056	0.954	0.953	1.739	1.735
$s = 4$	0.802	0.802	1.431	1.432	1.436	1.432	2.038	2.038	0.959	0.958	1.719	1.724
$s = 5$	0.805	0.805	1.426	1.426	1.426	1.426	2.029	2.029	0.965	0.964	1.717	1.720
$s = 6$	0.807	0.807	1.423	1.423	1.423	1.423	2.024	2.024	0.970	0.970	1.716	1.719
$s = 7$	0.810	0.810	1.421	1.421	1.421	1.421	2.022	2.022	0.975	0.975	1.717	1.718
$s = 8$	0.812	0.812	1.420	1.420	-	-	-	-	0.979	0.979	-	-
$s = 9$	0.813	0.813	-	-	-	-	-	-	-	-	-	-

Table 7: Example 6, $\epsilon = 0$: Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Example 7. $\mathbf{d = 3}$, $\mathbf{u \in W^{l,\bar{p}-\delta}(\Omega)}$, $\mathbf{l \in \{2, 3\}}$.

In this example, we investigate the behavior of the rates for $\Omega \subset \mathbb{R}^{d=3}$. We perform four groups of computations where for each group the values of the parameters are $\{\gamma = 0.2,\, l = 2,\, p = 1.2,\, \bar{p} \approx 1.667\}$, $\{\gamma = 1.2,\, l = 3,\, p = 1.2,\, \bar{p} \approx 1.667\}$, $\{\gamma = 0.75,\, l = 3,\, p = 1.2,\, \bar{p} \approx 1.333\}$ and finally $\{\gamma = 1.1,\, l = 3,\, p = 1.8,\, \bar{p} \approx 1.578\}$, see also the second row in Table 8. The expected convergence rates for $\gamma = 1.2$ and $\gamma = 1.1$, i.e., for the second and fourth group of computations, are calculated based on the estimate (4.14). The expected rate for $\gamma = 0.75$, i.e., third group, is given by (4.13) since in this case the inequality $l > 1 + d/\bar{p}$ does not hold, but the inequalities $l > d/\bar{p}$ and $\bar{p} \geq p$ are satisfied. Note that the parameters l and $\bar{p}$ for the first group corresponding to $\gamma = 0.2$ satisfy the conditions in the estimate (4.13). However, based on Proposition 5.2 we can easily see that for the particular solution $u = |x|^{\gamma}$ the following implication holds: If $1 < p \leq 2$ and $\gamma > (pq + 2q - 2d)/(pq)$, then $\mathbf{F}(\nabla u) \in [W^{1,q}(\Omega)]^d$. Thus, if $d = 3$, $q = 2$, $p = 1.2$ we obtain that $\mathbf{F}(\nabla u) \in [W^{1,2}(\Omega)]^d$ for $\gamma > 1 - 1/p \approx 0.1667$. Now, by Proposition 3.2 it follows that the expected convergence rate for $\gamma = 0.2$ is $r_D^F = r_I^F = 1$ for both polynomial degrees $k = 1$ and $k = 2$.

The numerically obtained convergence rates for the four groups are presented in the related columns in Table 8. They are in a very good agreement with the theoretically predicted rates given in Theorem 4.2. The numerical rates for $\gamma = 0.2$ are also in a good agreement with the estimates given in Proposition 3.2.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \lvert x\rvert^{\gamma}$, $\epsilon = 1$, $d = 3$, $h_0 = 0.4\sqrt{3}$										
h_0	$p = 1.2$, $\gamma = 0.2$, $l = 2$, $\bar{p} \approx 1.667$				$p = 1.2$, $\gamma = 1.2$, $l = 3$, $\bar{p} \approx 1.667$		$p = 1.2$, $\gamma = 0.75$, $l = 3$, $\bar{p} \approx 1.333$		$p = 1.8$, $\gamma = 1.1$, $l = 3$, $\bar{p} = 1.57895$	
$h_s = \frac{h_0}{2^s}$	$k = 1$		$k = 2$		$k = 2$		$k = 2$		$k = 2$	
	Expected rates r_D^F and r_I^F									
	$r_D^F = r_I^F = 1$				$r_D^F = r_I^F = 1.7$		$r_D^F = r_I^F \approx 1.333$		$r_D^F = r_I^F = 1.6$	
	Computed rates r_D^F and r_I^F									
	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F	r_D^F	r_I^F
$s = 0$	-	-	-	-	-	-	-	-	-	-
$s = 1$	0.639	1.267	1.119	1.229	1.864	1.914	1.535	1.679	1.790	1.877
$s = 2$	0.755	1.118	1.059	1.103	1.763	1.796	1.458	1.513	1.706	1.749
$s = 3$	0.798	1.040	1.021	1.047	1.710	1.727	1.401	1.423	1.654	1.674
$s = 4$	0.827	1.000	1.002	1.024	1.682	1.690	1.366	1.376	1.623	1.633
$s = 5$	0.850	0.979	0.992	1.016	1.669	1.674	1.347	1.353	1.607	1.613
$s = 6$	0.869	0.969	0.988	1.015	1.666	1.668	1.338	1.342	1.600	1.603
$s = 7$	0.886	0.966	-	-	-	-	-	-	-	-

Table 8: Example 7, $d = 3$: Convergence rates r_D^F and r_I^F for the errors $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_{h_s})\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_{h_s}^k u)\|_{L^2(\Omega)}$.

Example 8. ***Convergence rates for*** $|\mathbf{u} - \mathbf{u_h}|_{\mathbf{W^{1,p}(\Omega)}}$, $\mathbf{p = 1.5}$, $\bar{\mathbf{p}} \approx \mathbf{1.538}$.

Below, we perform several computations for investigating the convergence behavior of the error $|u - u_h|_{W^{1,p}(\Omega)}$ and the corresponding interpolation error $\left|u - I_h^k u\right|_{W^{1,p}(\Omega)}$. The computatioanl rates are denoted by $r_D^{1,p}$ and $r_I^{1,p}$, respectively, and are computed using similar formulas as those in (5.1) with $|u - u_h|_{W^{1,p}(\Omega)}$ and $\left|u - I_h^k u\right|_{W^{1,p}(\Omega)}$. While the convergence rates given in Theorem 4.2 agree with the numerically obtained ones for $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)\|_{L^2(\Omega)}$ and $\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla I_h^k u)\|_{L^2(\Omega)}$ in all the examples above, the rates for $|u - u_h|_{W^{1,p}(\Omega)}$ and $\left|u - I_h^k u\right|_{W^{1,p}(\Omega)}$ seem to be always higher. The theoretical justification of this observation requires further investigation. During this example we perform three groups of computations where the values of the parameters for each group are $\{\gamma = 0.7, l = 2, p = 1.5, \bar{p} = 1.538\}$, $\{\gamma = 1.7, l = 3, p = 1.5, \bar{p} = 1.538\}$ and $\{\gamma = 2.7, l = 4, p = 1.5, \bar{p} = 1.538\}$ correspondingly. The theoretically predicted rates for these tests are given in Theorem 4.2. For each computation the problem is solved using polynomial degree $k \geq l - 1$, and thus we expect that the related convergence rates are determined by the regularity of the solution. The numerical results are shown in Table 9. We observe that for each case the convergence rates $r_D^{1,p}$ of the solution are very close with the corresponding interpolation rates $r_I^{1,p}$, i.e., the rates for the quantity $\left|u - I_{h_s}^k u\right|_{W^{1,p}(\Omega)}$. Anyway, in any test case we can observe that the rates are higher than the theoretically predicted rates given in Theorem 4.2. We have observed the same behavior, i.e., higher convergence rates than those for the error measured in the **F**-quasinorm (given by Theorem 4.2) for all of the presented tests above.

$u \in W^{l,\bar{p}-\delta}(\Omega)$ with $u = \lvert x \rvert^{\gamma}$, $\epsilon = 1$, $p = 1.5$, $d = 2$, $h_0 = 0.1\sqrt{2}$												
h_0	$\gamma = 0.7$ $l = 2$ $\bar{p} \approx 1.538$						$\gamma = 1.7$ $l = 3$ $\bar{p} \approx 1.538$				$\gamma = 2.7$ $l = 4$ $\bar{p} \approx 1.538$	
$h_s = \frac{h_0}{2^s}$	$k = 1$		$k = 2$		$k = 3$		$k = 2$		$k = 3$		$k = 3$	
	Computed rates $r_D^{1,p}$ and $r_I^{1,p}$											
	$r_D^{1,p}$	$r_I^{1,p}$	$r_D^{1,p}$	$r_I^{1,p}$	$r_D^{1,p}$	$r_I^{1,p}$	$r_D^{1,p}$	$r_I^{1,p}$	$r_D^{1,p}$	$r_I^{1,p}$	$r_D^{1,p}$	$r_I^{1,p}$
$s = 0$	-	-	-	-	-	-	-	-	-	-	-	-
$s = 1$	0.840	0.901	1.090	1.099	1.105	1.107	1.974	1.971	2.187	2.165	3.020	3.006
$s = 2$	0.858	0.898	1.063	1.067	1.068	1.070	1.935	1.933	2.103	2.101	2.944	2.945
$s = 3$	0.874	0.903	1.047	1.050	1.049	1.051	1.921	1.920	2.069	2.068	2.920	2.920
$s = 4$	0.889	0.910	1.039	1.042	1.040	1.042	1.919	1.918	2.051	2.051	2.914	2.914
$s = 5$	0.901	0.918	1.035	1.037	1.036	1.037	1.922	1.922	2.042	2.042	2.915	2.916
$s = 6$	0.912	0.925	1.033	1.035	1.033	1.035	1.928	1.927	2.038	2.037	2.891	2.921
$s = 7$	0.921	0.932	1.032	1.034	1.032	1.034	1.933	1.933	2.035	2.035	2.900	2.927
$s = 8$	0.929	0.938	-	-	-	-	1.939	1.939	-	-	-	-
$s = 9$	0.936	0.944	-	-	-	-	1.943	1.944	-	-	-	-

Table 9: Example 8: Convergence rates $r_D^{1,p}$ and $r_I^{1,p}$ for the errors $\lvert u - u_{h_s} \rvert_{W^{1,p}(\Omega)}$ and $\left\lvert u - I_{h_s}^k u \right\rvert_{W^{1,p}(\Omega)}$.

6 Conclusions

In this work, existence and uniqueness results for a class of quasilinear elliptic problems have been shown and an a priori error analysis for finite element discretizations has been derived. The error analysis utilizes the same quasinorm quantities which are introduced in [21] and made a step forward with respect to the interpolation and the discretization estimates. The technique used in our analysis relies on approximation error estimates in standard Sobolev $W^{1,p}$-seminorms and therefore can be applied to other finite element spaces for which such approximation results are available. In particular, our analysis applies to continuous piecewise polynomial spaces of higher order where the regularity assumptions are posed directly on the solution u itself. The analysis showed that optimal approximation estimates in the associated $\mathbf{F}$-quasinorm can be also obtained for solutions u having higher regularity, i.e., $u \in W^{l,\bar{p}}(\Omega)$ with $l \geq 2$ and $\bar{p} \geq 2$ in conjunction with $\mathbb{P}_k$ finite element spaces satisfying $k = l - 1$. It is worth pointing out that these convergence rates do not depend on the parameter p in Assumption 2, but only on the regularity of the solution u. The theoretical error estimates have been confirmed by performing a series of numerical tests. The numerical rates and theoretically predicted rates were in a very good agreement. Our numerical experiments also indicate that the regularity conditions on u that guarantee optimal convergence rates cannot be relaxed. More precisely, they show that for the case $1 < p \leq 2$ (with p being the parameter in Assumption 2) if u is not in $W^{l,2}(\Omega)$, then the convergence rates can be strictly less than $l - 1$ for polynomial spaces $\mathbb{P}_k$ with $k = l - 1$. On the other hand, the rates for the error $\lvert u - u_h \rvert_{W^{1,p}(\Omega)}$ and $\left\lvert u - I_h^k u \right\rvert_{W^{1,p}(\Omega)}$ were found to be always higher than the corresponding theoretically predicted and numerically validated rates for the $\mathbf{F}$-quasinorm. This is a point which needs further investigation. Also, further work could be to investigate the relation between the $\mathbf{F}$-quasinorm and other quasinorms that can be inferred from functional type duality methods which have been proposed in the literature. The derivation of a discretization error analysis for more general quasilinear elliptic problems by following similar ideas seems to be also feasible. These last subjects are going to be discussed by the authors in forthcoming works.

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