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On Inner Calmness*, Generalized Calculus, and Derivatives of the Normal-cone Map

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On inner calmness*, generalized calculus, and derivatives of the normal-cone map

Matúš Benko*

Abstract. In this paper, we study the inner-type continuity and Lipschitzian properties of set-valued maps. We introduce the new notion of inner calmness* and show its remarkable features, in particular that it is satisfied by polyhedral maps. Then we utilize these inner-type conditions to extend the known and to build new generalized differential calculus rules, focusing on the primal objects (e.g. tangent cones). The exact chain rule for the graphical derivatives deserves the special attention. Finally, we apply these results to compute or estimate the generalized derivatives of the so-called normal cone mapping, that are critical for stability analysis, etc. As a specific application, we derive some interesting results regarding the newly developed property of semismoothness* of the normal cone maps.

Key words. inner calmness*, generalized differential calculus, tangents to image sets, chain rule, normal-cone mapping, polyhedral multifunction

AMS Subject classification. 49J53, 49J52, 90C31

1 Introduction

The so-called *normal cone mapping*, closely related to variational inequalities, is a standard tool of variational analysis that plays the crucial role in the study of various stability and sensitivity properties of the solutions of mathematical programs as well as in the formulation of optimality conditions for mathematical programs with equilibrium constraints. For more information about the normal cone mapping and related issues we refer to the standard textbooks [4, 8, 23, 28, 29, 34]. Recently a lot of success has been achieved in computation of the graphical derivative of the normal cone mapping, see, e.g., [3, 5, 14, 15, 17, 19]. These results were obtained by a direct approach, i.e., without relying on the generalized calculus rules. Apart from the obvious fact that the direct approach may be more flexible in situations where the calculus falls short, there is yet another basic limitation, namely the lack of a suitable calculus for primal objects, such as the graphical derivative based on the tangent cone.

Observing the above results and their proofs, we ask ourselves the question, whether there are some underlying calculus principles in play and whether we can properly understand and identify them and bring them to the light. Not only we can achieve this and propose improved as well as new calculus rules, we also discover a new property of *inner calmness**, which turns out to be the most essential element of this calculus and it also seems to be very interesting on its own.

The inner-type conditions, be it the inner semicontinuity or some inner Lipschitzian property, are considerably underdeveloped compared to their outer counterparts, such as the outer semicontinuity

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and the calmness. Especially the calmness and its inverse equivalent, metric subregularity, have already been extensively studied by many prominent researcher, see [7, 8, 9, 10, 11, 20, 22, 23, 24, 29, 31] and the references therein. In this paper, the inner-type conditions are at the center of attention. In particular, in Section 3 we make the strong case for the inner semicompactness and the inner calmness*, by showing that they are satisfied in for certain general multiplier map. Moreover, we show that *polyhedral set-valued maps enjoy the inner calmness* property*, hence providing an inner counterpart to the famous result on calmness (upper Lipschitzness) of polyhedral maps by Robinson [32].

The generalized differential calculus belongs among the most fundamental tools of modern variational analysis [23, 26, 28, 29, 31, 34]. In calculus, there are two main principles, which we will refer to as "forward" and "backward". The "forward" principle corresponds to the image rule, i.e., when the set under investigation is generated as an image of another given set under a given map ($Q := \varphi(C)$). The "backward" principle corresponds to the pre-image rule, i.e., when the set under investigation is generated as a pre-image of another given set under a given map ($C := \varphi^{-1}(Q)$). The image and pre-image rule form the base of the whole calculus and almost all the standard calculus rules for sets, functions and set-valued maps can be derived from these two results. Of course, one can choose a different set of results as a base. It is very common to start with (extended real-valued) functions.

The backward principle seems to be better understood and more worked out and is closely related to a condition of the (better known) outer-type. Indeed, the main assumption for the backward results, typically called qualification condition, is known to be the calmness of the associated perturbation mapping, often equivalently expressed via the metric subregularity of the feasibility mapping [20, 24]. In Section 4, we propose comprehensive forward calculus rules, where the inner-type properties play the key role.

The calculus rules are divided from yet another important perspective, namely whether they deal with the primal or the dual objects. Hence, the most studied and, arguably, the most important, are in fact backward rules for dual objects, while the backward rules for primal object are in turn the simplest ones. Indeed, in the pre-image rule for tangent cones [34, Theorem 6.31], one inclusion is always valid, while the other one also holds under the calmness [21, Proposition 1]. It is also well-known, that the inner semicontinuity and inner semicompactness yield the forward rules for dual objects [25]. In order to obtain the reasonable forward rules for primal objects, the least explored area, however, one needs to strengthen these inner notion to suitable inner Lipschitzian conditions. This is precisely where the inner calmness* comes in handy.

By filling the gap of forward rules for primal objects, we derive the particularly interesting *chain rule for the graphical derivatives* in the exact form.

Note also that there is the obvious connection between the tangent cones and *directional* limiting normal cones. Hence, this paper is heavily based on the material from [2], where a comprehensive calculus for directional limiting constructions was developed.

In Section 5, as an application we return to the normal cone mapping, where we compute or estimate its generalized derivatives. In particular, the computation of the graphical derivative is just an elegant and simple corollary of our approach, which works very robustly in three consider models. On the other hand, we also see the limitations of our calculus-based approach compared to the results obtained by the direct computations.

Interestingly, in Section 5 there is no mentioning of the inner calmness*, since the results are obtained as a combination of the Theorem 3.10 from Section 3 (where the inner calmness* is shown to holds for certain multiplier map) and the calculus from Section 4 (where the inner calmness* acts as an assumption). In other words, the inner calmness* appears only in the intermediate step, which also verifies that the inner calmness* was always there, just hidden.

The following notation is employed. Given a set $A \subset \mathbb{R}^n$, the closure and interior of A are denoted, respectively, by $\mathrm{cl}A$ and $\mathrm{int}A$, $\mathrm{sp}A$ stands for the linear hull of A and A° is the (negative) polar of A. We denote by $\mathrm{dist}(\cdot,A) := \inf_{y\in A}\|\cdot - y\|$ the usual point to set distance with the convention $\mathrm{dist}(\cdot,\emptyset)=\infty$ and $\langle\cdot,\cdot\rangle$ stands for the standard scalar product. Further, $\mathscr{B}$, $\mathscr{S}$ stand respectively the closed unit ball and the unit sphere of the space in question. Given a vector $a\in\mathbb{R}^n$, $[a]$ is the linear space generated by a and $[a]^\perp$ stands for the orthogonal complement to $[a]$. Given a (sufficiently) smooth function $f:\mathbb{R}^n\to\mathbb{R}$, denote its gradient and Hessian at x by $\nabla f(x)$ and $\nabla^2 f(x)$, respectively. Considering further a vector function $\varphi:\mathbb{R}^n\to\mathbb{R}^s$ with $s>1$, denote by $\nabla\varphi(x)$ the Jacobian of φ at x, i.e., the mapping $x\to\nabla\varphi(x)$ goes from $\mathbb{R}^n$ into the space of $s\times n$ matrices, denoted by $(\mathbb{R}^s)^n$. Moreover, for a mapping $\beta:\mathbb{R}^n\to(\mathbb{R}^s)^m$ and a vector $y\in\mathbb{R}^s$, we introduce the scalarized map $\langle y,\beta\rangle:\mathbb{R}^n\to\mathbb{R}^m$ given by $\langle y,\beta\rangle(x)=\beta(x)^T y$. Finally, following traditional patterns, we denote by $o(t)$ for $t\ge 0$ a term with the property that $o(t)/t\to 0$ when $t\downarrow 0$.

2 Preliminaries

We begin by recalling several definitions and results from variational analysis. Let $\Omega\subset\mathbb{R}^n$ be an arbitrary closed set and $\bar{x}\in\Omega$. The *tangent* (also called *Bouligand* or *contingent*) *cone* to Ω at $\bar{x}$ is given by

$$T_\Omega(\bar{x}) := \{u\in\mathbb{R}^n \mid \exists (t_k)\downarrow 0, (u_k)\to u : \bar{x}+t_k u_k\in\Omega\ \forall k\}.$$

We denote by

$$\widehat{N}_\Omega(\bar{x}) := T_\Omega(\bar{x})^\circ = \{x^*\in\mathbb{R}^n \mid \langle x^*,u\rangle\le 0\ \forall u\in T_\Omega(\bar{x})\}$$

the *regular* (*Fréchet*) *normal cone* to Ω at $\bar{x}$. The *limiting* (*Mordukhovich*) *normal cone* to Ω at $\bar{x}$ is defined by

$$N_\Omega(\bar{x}) := \{x^*\in\mathbb{R}^n \mid \exists (x_k)\to\bar{x},\ (x_k^*)\to x^* : x_k^*\in\widehat{N}_\Omega(x_k)\ \forall k\}.$$

Finally, given a direction $u\in\mathbb{R}^n$, we denote by

$$N_\Omega(\bar{x};u) := \{x^*\in\mathbb{R}^n \mid \exists (t_k)\downarrow 0, (u_k)\to u, (x_k^*)\to x^* : x_k^*\in\widehat{N}_\Omega(\bar{x}+t_k u_k)\ \forall k\}$$

the *directional limiting normal cone* to Ω *in direction* u at $\bar{x}$.

If $\bar{x}\notin\Omega$, we put $T_\Omega(\bar{x})=\emptyset$, $\widehat{N}_\Omega(\bar{x})=\emptyset$, $N_\Omega(\bar{x})=\emptyset$ and $N_\Omega(\bar{x};u)=\emptyset$. Further note that $N_\Omega(\bar{x};u)=\emptyset$ whenever $u\notin T_\Omega(\bar{x})$. If Ω is convex, then $\widehat{N}_\Omega(\bar{x})=N_\Omega(\bar{x})$ amounts to the classical normal cone in the sense of convex analysis and we will write $N_\Omega(\bar{x})$. More generaly, we say that Ω is *Clarke regular* at point $\bar{x}$, provided $\widehat{N}_\Omega(\bar{x})=N_\Omega(\bar{x})$ and we write $N_\Omega(\bar{x})$ in such case.

The following generalized derivatives of set-valued mappings are defined by means of the tangent cone and the normal cone to the graph of the mapping. Let $M:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ be a set-valued mapping having locally closed graph around $(\bar{x},\bar{y})\in\mathrm{gph}M := \{(x,y)\mid y\in M(x)\}$. The set-valued map $DM(\bar{x},\bar{y}):\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, defined by

$$DM(\bar{x},\bar{y})(u) := \{v\in\mathbb{R}^m \mid (u,v)\in T_{\mathrm{gph}M}(\bar{x},\bar{y})\}, u\in\mathbb{R}^n$$

is called the *graphical derivative* of M at $(\bar{x},\bar{y})$. The set-valued map $\widehat{D}^*M(\bar{x},\bar{y}):\mathbb{R}^m\rightrightarrows\mathbb{R}^n$

$$\widehat{D}^*M(\bar{x},\bar{y})(v^*) := \{u^*\in\mathbb{R}^n \mid (u^*,-v^*)\in\widehat{N}_{\mathrm{gph}M}(\bar{x},\bar{y})\}, v^*\in\mathbb{R}^m$$

is called the *regular (Fréchet) coderivative* of M at $(\bar{x},\bar{y})$. The map $D^*M(\bar{x},\bar{y}):\mathbb{R}^m\rightrightarrows\mathbb{R}^n$

$$D^*M(\bar{x},\bar{y})(v^*) := \{u^*\in\mathbb{R}^n \mid (u^*,-v^*)\in N_{\mathrm{gph}M}(\bar{x},\bar{y})\}, v^*\in\mathbb{R}^m$$

is called the *limiting (Mordukhovich) coderivative* of M at $(\bar{x},\bar{y})$. Given a pair of directions $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$, the map $D^*M((\bar{x},\bar{y});(u,v)) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, given by

$$D^*M((\bar{x},\bar{y});(u,v))(v^*) := \{u^* \in \mathbb{R}^n \mid (u^*,-v^*) \in N_{\mathrm{gph}\,M}((\bar{x},\bar{y});(u,v))\}, v^* \in \mathbb{R}^m$$

is called the *directional limiting coderivative* of M *in direction* (u,v) at $(\bar{x},\bar{y})$.

We briefly sum up some basic results from variational analysis of polyhedral sets. Recall that a set $D \subset \mathbb{R}^s$ is said to be *convex polyhedral* if it can be represented as the intersection of finitely many halfspaces. We say that a set $E \subset \mathbb{R}^s$ is *polyhedral* if it is the union of finitely many convex polyhedral sets. A set-valued map is called (convex) polyhedral, if its graph is a (convex) polyhedral set.

If a set E is polyhedral, then for every $\bar{z} \in E$ there is a neighborhood W of $\bar{z}$ such that

$$(E-\bar{z}) \cap W = T_E(\bar{z}) \cap W.$$

Given a convex polyhedral set D and a point $\bar{z} \in D$, the tangent cone $T_D(\bar{z})$ and the normal cone $N_D(\bar{z})$ are convex polyhedral cones and there is a neighborhood W of $\bar{z}$ such that

$$T_D(z) = T_D(\bar{z}) + [z-\bar{z}] \supset T_D(\bar{z}), \;\; N_D(z) = N_D(\bar{z}) \cap [z-\bar{z}]^{\perp} \subset N_D(\bar{z}) \;\; \forall z \in W.$$

The graph of the normal cone mapping to D is a polyhedral set and for every pair $(z,z^*) \in \mathrm{gph}\,N_D$ the *reduction lemma* [8, Lemma 2E.4] yields

$$T_{\mathrm{gph}\,N_D}(z,z^*) = \mathrm{gph}\,N_{\mathscr{K}_D(z,z^*)}, \tag{2.1}$$

where $\mathscr{K}_D(z,z^*) := T_D(z) \cap [z^*]^{\perp}$ stands for the *critical cone* to D at (z,z^*) (this definition is not restricted to polyhedral sets).

For two convex polyhedral cones $K_1, K_2 \subset \mathbb{R}^s$ their polars as well as their sum K_1+K_2 and their intersection $K_1 \cap K_2$ are again convex polyhedral cones and

$$(K_1+K_2)^{\circ} = K_1^{\circ} \cap K_2^{\circ}, \; (K_1 \cap K_2)^{\circ} = K_1^{\circ} + K_2^{\circ}.$$

For a convex polyhedral cone $K \subset \mathbb{R}^s$ and a point $z \in K$ we have

$$T_K(z) = K + [z], \; N_K(z) = K^{\circ} \cap [z]^{\perp}.$$

A face $\mathscr{F}$ of K can always be written in the form $\mathscr{F} = K \cap [z^*]^{\perp}$ for some $z^* \in K^{\circ}$ and there is only finitely many faces of convex polyhedral cones.

In the rest of the section, we recall some well-known properties of set-valued maps. Before introducing the Lipschitzian conditions, which, admittedly, are of the prime interest, we begin with some continuity notions. For set-valued maps, continuity is composed of two semicontinuities, *inner* and *outer*. First, we want to briefly discuss some ideas and principles related to this inner and outer paradigm, which later extends to the Lipschitzian properties. For simplicity, we explain these ideas in slightly informal manner. Given a mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, continuity of S at $\bar{y} \in \mathbb{R}^m$ is related to the following condition. For every $\rho, \varepsilon > 0$ there is a neighborhood V of $\bar{y}$ such that

$$S(y') \cap \rho\mathscr{B} \subset S(y) + \varepsilon\mathscr{B} \quad \forall y', y \in V.$$

With respect to the above estimate let us call y the *estimating* and y' the *estimated* variable. Then, the outer semicontinuity is related to the same estimate with the estimating variable y being fixed to $\bar{y}$, while the inner semicontinuity is related to the estimate with the estimated variable y' being fixed to $\bar{y}$. We refer to [34, Proposition 5.12, Exercise 5.13] for the precise results.

The outer semicontinuity seems to be easier to deal with and better understood and this issue is even more obvious when comparing the well-known and heavily utilized "outer-Lipschitzian" notions with the quite undeveloped "inner-Lipschitzian" ones. It turns out, however, that the inner paradigm, related to fixing the estimated variable y', is also very important and interesting. In this paper, we will mainly work with four properties of inner-type.

Let us proceed with the exact definitions. Recall that S is *inner semicompact at $\bar{y}$ with respect to (wrt) $\Omega \subset \mathbb{R}^m$* if for every sequence $y_k \to \bar{y}$ with $y_k \in \Omega$ there exists a subsequence K of $\mathbb{N}$ and a sequence $(x_k)_{k \in K}$ with $x_k \in S(y_k)$ for $k \in K$ converging to some $\bar{x}$. Given $\bar{x} \in S(\bar{y})$, we say that S is *inner semicontinuous at $(\bar{y}, \bar{x})$ wrt Ω* if for every sequence $y_k \to \bar{y}$ with $y_k \in \Omega$ there exists a sequence $x_k \in S(y_k)$ with $x_k \to \bar{x}$. If $\Omega = \mathbb{R}^m$, we speak only about inner semicompactness at $\bar{y}$ and inner semicontinuity at $(\bar{y}, \bar{x})$. For more details regarding these standard notions we refer to [28].

We point out the inner semicompactness is implied by the simpler local boundedness condition. In fact, the local boundedness is often imposed in the development of the calculus and the inner semicompactness is avoided, see [34]. We believe, however, that the boundedness assumption is slightly misleading, since its purpose is to *restrict* the map, while the inner semicompactness, in a sense, says the opposite - be as unbounded as you like, I just need a convergent subsequence. In Section 3, we will see the impact of these differences when dealing with the multiplier map.

Moreover, we stress that the inner semicontinuity from [34] is considered not at point $(\bar{y}, \bar{x})$ but only at $\bar{y}$ and it actually corresponds to the requirement that S is inner semicontinuous at $(\bar{y}, \bar{x})$ for every $\bar{x} \in S(\bar{y})$, which is obviously stronger condition.

In [2], we needed to strengthen these properties by controlling the rate of convergence $x_k \to \bar{x}$. To this end, we introduced the *inner calmness* by building upon the inner semicontinuity, namely, we called *S inner calm at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ wrt Ω* if there exist $\kappa > 0$ and a neighborhood V of $\bar{y}$ such that

$$\bar{x} \in S(y) + \kappa \|y - \bar{y}\| \mathscr{B} \quad \forall y \in V \cap \Omega,$$

or, equivalently, if there exist $\kappa > 0$ such that for every sequence $y_k \to \bar{y}$ with $y_k \in \Omega$ there exists a sequence $x_k \in S(y_k)$ with

$$\|x_k - \bar{x}\| \le \kappa \|y_k - \bar{y}\|. \tag{2.2}$$

In the literature, this property can be found also under several other names, such as, e.g., the Lipschitz lower semicontinuity [27] or the recession with linear rate [6, 23]. In particular, [6] contains a comprehensive study of this property.

Naturally, one can also define the stronger version of inner calmness *at $\bar{y}$*, based on the inner semicontinuity at $\bar{y}$. Taking into account that even the inner calmness at $(\bar{y}, \bar{x})$ is often too restrictive, we follow the opposite approach and introduce a milder concept of inner calmness based on the inner semicompactness.

Definition 2.1. *A set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is called* inner calm* at $\bar{y} \in \mathbb{R}^m$ wrt $\Omega \subset \mathbb{R}^m$ *if there exists $\kappa > 0$ such that for every sequence $y_k \to \bar{y}$ with $y_k \in \Omega$, there exist a subsequence K of $\mathbb{N}$, together with a sequence $(x_k)_{k \in K}$ and $\bar{x} \in \mathbb{R}^n$ with $x_k \in S(y_k)$ for $k \in K$ and* (2.2)*. Moreover, we say that S is inner calm* at $\bar{y}$ wrt Ω* in the fuzzy sense*, provided $x_k \in S(y_k + o(\|y_k - \bar{y}\|))$ instead of $x_k \in S(y_k)$.*

Clearly, the inner calmness at $(\bar{y}, \bar{x})$ implies both, the inner calmness* at $\bar{y}$ as well as the inner semicontinuity at $(\bar{y}, \bar{x})$, and either of the two implie the inner semicompactness at $\bar{y}$.

Note that each of the four inner conditions implies that $S(y) \neq \emptyset$ for $y \in \Omega$ near $\bar{y}$. While this can be desirable in some situations, it can also be quite restrictive. For our purposes, however, we will often consider these properties wrt to the *domain* of map S, given by $\operatorname{dom} S := \{y \in \mathbb{R}^m \mid S(y) \neq \emptyset\}$, adding no restriction at all.

Next we briefly introduce one important outer Lipschitzian notion. We say that S is *calm* at $(\bar{y},\bar{x}) \in \operatorname{gph} S$, provided there exist $\kappa > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$S(y) \cap U \subset S(\bar{y}) + \kappa\|y-\bar{y}\|\mathscr{B} \quad \forall y \in V. \tag{2.3}$$

Interestingly, in the definition of calmness, the crucial neighborhood is U, not V. Indeed, neighborhood U can be reduced (if necessary) in such a way that neighborhood V can be replaced by the whole space $\mathbb{R}^m$, cf. [8, Exercise 3H.4]. This is also related to the well-known fact that calmness of S at $(\bar{y},\bar{x})$ is equivalent to the metric subregularity of $M := S^{-1}$ at $(\bar{x},\bar{y})$.

Definition 2.2. *Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $(\bar{x},\bar{y}) \in \operatorname{gph} M$. We say that M is* metrically subregular *at $(\bar{x},\bar{y})$ provided there exist $\kappa > 0$ and a neighborhood U of $\bar{x}$ such that*

$$\operatorname{dist}(x, M^{-1}(\bar{y})) \leq \kappa \operatorname{dist}(\bar{y}, M(x)) \quad \forall x \in U.$$

In case of a single-valued mapping $\varphi : \mathbb{R}^m \to \mathbb{R}^n$, the inner calmness at $(\bar{y},\varphi(\bar{y}))$ reads as

$$\|\varphi(y) - \varphi(\bar{y})\| \leq \kappa\|y-\bar{y}\| \quad \forall y \in V. \tag{2.4}$$

We will refer to this property as calmness of φ at $\bar{y}$. Quite surprisingly, however, neither the calmness definition from (2.3) nor the inner calmness* yield (2.4)! This is due to the neighborhood U in case of calmness and due to not requiring $\bar{x} \in S(\bar{y})$ in the definition of inner calmness*. Indeed, $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(y) = \begin{cases} 0 & \text{for } y \leq 0, \\ 1 & \text{for } y > 0 \end{cases}$$

is clearly calm at $(\bar{y},\varphi(\bar{y})) = (0,0)$ by (2.3) as well as inner calm* at $\bar{y}$, despite being discontinuous at $\bar{y}$. Naturally, if we restrict ourselves to continuous mappings, all three notions coincide and read as (2.4). Further, φ is called *Lipschitz continuous near $\bar{y}$* if the inequality

$$\|\varphi(y) - \varphi(y')\| \leq \kappa\|y-y'\| \; \forall y,y' \in V$$

is fulfilled with $\kappa > 0$ and V being a neighborhood of $\bar{y}$.

Finally, we will make use of the *directional* counterparts of the above properties. For the purpose of this paper, it is sufficient to say that a sequence $y_k \in \mathbb{R}^m$ converges to some $\bar{y}$ from direction $v \in \mathbb{R}^m$ if there exist $t_k \downarrow 0$ and $v_k \to v$ with $y_k = \bar{y} + t_k v_k$. Then, the directional versions of the above properties are obtain by replacing the standard converge by the converge from a prescribed direction v.

Note that the standard approach is to define the directional properties by means of the directional neighborhoods, see [10] for the details.

3 Inner semicompactness and inner calmness*

In this section, we provide some remarkable results regarding our newly introduced concept of inner calmness*. More precisely, we discuss some interesting cases when the inner calmness* is satisfied, while its role as an assumption is highlighted in the next section dealing with the calculus rules. We point out that some principles can be explained in terms of the inner semicompactness. Then, adding certain polyhedrality assumptions allows us to control the rate of convergence and hence leads to the inner calmness*. From a different angle, we show that the inner semicompactness and the inner calmness* are very suitable relaxations of quite restrictive properties of inner semicontinuity and inner calmness.

3.1 Polyhedral set-valued maps

We begin by the simple example showing the limitations of inner semicontinuity and inner calmness.

Example 3.1. *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}$ by given by*

$$S(y) = \begin{cases} 0 & \text{for } y \leq 0, \\ 1 & \text{for } y \geq 0. \end{cases}$$

It is easy to see that S is not inner semicontinuous at $(\bar{y},\bar{x}) = (0,0)$, due to $y_k := 1/k \to \bar{y}$ and $S(y_k) = 1 \not\to 0$, or at $(\bar{y},\bar{x}) = (0,1)$, due to $y_k := -1/k \to \bar{y}$ and $S(y_k) = 0 \not\to 1$.

On the other hand, S is clearly inner semicompact (even inner calm) at $\bar{y}$. Indeed, given a sequence $y_k \to \bar{y}$, we can* choose *$\bar{x}$ to be either* 0 *or* 1*, depending on which of the two values is attained by $S(y_k)$ infinitely many times.*

The above example shows that the lack of convexity of the graph can easily lead to the violation of this property. Let us now look at maps S with the graph having a special structure. More precisely, consider a map $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with its graph being a finite union of closed sets G_i for $i = 1,\dots,l$ and denote by S_i the maps with $\operatorname{gph} S_i = G_i$, referred to as the components of S.

Note that since the finite union of closed sets remains a closed set, so is the graph of S. This, in turn, is equivalent to S being outer semicontinuous everywhere, see [34, Theorem 5.7(a)]. The closedness of $\operatorname{gph} S$ is not very restrictive and often needed in order to work with generalized derivatives of S. This also suggests that the outer semicontinuity is easier to handle than its inner counterpart.

We will now show that the property of inner semicompactness is also preserved under the finite unions and hence serves as a “completion” of the inner semicontinuity with respect to finite unions.

Lemma 3.2. *Given a map $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, assume that its components S_i are inner semicompact at $\bar{y} \in \mathbb{R}^m$ wrt to* $\operatorname{dom} S_i$*. Then S is inner semicompact at $\bar{y}$ wrt to its domain.*

Proof. If $\bar{y} \notin \operatorname{cl}(\operatorname{dom} S)$, the inner semicompactness follows. Hence, let $y_k \to \bar{y}$ with $y_k \in \operatorname{dom} S$. By passing to a subsequence if necessary, we may assume that there exists i with $y_k \in \operatorname{dom} S_i$ for all k. The inner semicompactness of S_i now yields the existence of a sequence $x_k \in S_i(y_k)$ converging to some $\bar{x}$. Since $x_k \in S(y_k)$, the claim follows. □

Note that we did not have to require the domains to be closed, since in the definition of inner semicompactness we do not impose $\bar{y}$ to be in the domain.

Clearly, we can strengthen the assumptions to inner semicontinuity, but we only need to ask for the inner semicontinuity of those components S_i with $\bar{y} \in \operatorname{cl}(\operatorname{dom} S_i)$. For the other components the inner semicompactness follows for free.

Recalling the classical result [34, Theorem 5.9] on inner semicontinuity from the graph-convexity, one may wonder whether it is possible to derive the inner semicompactness of maps with the graph being the finite union of convex sets. Since [34, Theorem 5.9] yields the inner semicontinuity at every point of the interior of domain, we readily obtain the following statement.

Corollary 3.3. *Assume that gphS can be written as a finite union of convex sets and $\bar{y} \in \operatorname{int}(\operatorname{dom} S_i)$ for some i. Then S is inner semicompact at $\bar{y}$.*

Note that S does not have to be inner semicompact at every $\bar{y} \in \operatorname{int}(\operatorname{dom} S)$, since such $\bar{y}$ may lie on the boundaries of several domains of S_i.

Next we will show that, unless special assumptions are made, it is not easy to extend the inner semicontinuity, or even inner semicompactness, to the boundary of domain of a graph-convex map.

Note that a possible way to achieve such an extension can perhaps be to follow the approach from [33, Theorem 10.2] on closely related issue of upper semicontinuity of convex functions, see also [34, Example 5.5] for the connection.

Here we instead propose an example showing that even the map with closed convex graph and closed domain may not be inner semicompact wrt its domain at every point.

Example 3.4. *Consider the set G in $\mathbb{R}^2 \times \mathbb{R}$ given by the union of point $(\bar{y},0) = (\bar{y}_1,\bar{y}_2,0) = (1,0,0)$ and set*

$$y_1 = \cos(t),\ y_2 = \sin(t),\ x = \frac{1}{t} \quad \text{for } t \in (0,2\pi). \tag{3.5}$$

Now let $S : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ be the set-valued map with the graph being the closure of the convex hull of G. Note also that the domain of S is the closed unit ball. We claim that S is not inner semicompact at $\bar{y}$ wrt to $\operatorname{dom} S$*. Indeed, consider a sequence $t_k \downarrow 0$ and set $y_k := (\cos(t_k),\sin(t_k)) \to \bar{y}$. We will show that there is no $x_k \in S(y_k)$ with $x_k < 1/t_k \to \infty$. To this end, for every k we construct a halfspace containing set G but no (y_k,x_k) with $x_k < 1/t_k$. Set*

$$b_k := \left(1, \frac{1-\cos(t_k)}{\sin(t_k)} + \frac{t_k(1-\cos(t_k))}{\sin(t_k)(t_k\cos(t_k)+\sin(t_k))}, \frac{-t_k^2(1-\cos(t_k))}{t_k\cos(t_k)+\sin(t_k)}\right)$$

and consider the halfspace $\mathscr{H}_k := \{(y,x) \mid \langle b_k,(y,x)\rangle \le 1\}$. Clearly, $(1,0,0) \in \mathscr{H}_k$. Moreover, every point (y_1,y_2,x) of the form (3.5) *also belongs to $\mathscr{H}_k$, since the function*

$$t \to \langle b_k,(\cos(t),\sin(t),1/t)\rangle$$

attains the global maximum over $(0,2\pi)$ at $t = t_k$ with value 1*. Finally, a point (y_k,x) belongs to $\mathscr{H}_k$ if and only if $x \ge 1/t_k$. Hence, taking into account that* $\operatorname{gph} S$ *is the intersection of all the closed halfspaces containing G by [33, Corollary 11.5.1], there is no $x_k \in S(y_k)$ with $x_k < 1/t_k$. Hence, S is not inner semicompact at $\bar{y}$ wrt to* $\operatorname{dom} S$.

Next, we turn our attention to the corresponding Lipschitzian notions. It is not surprising that the analogous principles hold true here, where the only difference is that we control the rate of convergence.

Lemma 3.5. *Given a map $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with* $\operatorname{gph} S = \bigcup G_i$ *for closed sets G_i, assume that its components S_i are inner calm* at $\bar{y} \in \mathbb{R}^m$ wrt to* $\operatorname{dom} S_i$*. Then S is inner calm* at $\bar{y}$ wrt to its domain.*

The proof follows by the same arguments as the proof of the previous lemma, with the constant κ of inner calmness* of S given as the maximum of the constants κ_i of inner calmness* of S_i.

The above lemma allows us to establish the main result of this section regarding polyhedral maps. In the polyhedral setting, there is no problem with the points on the boundary of the domain. Indeed, the prominent result of Walkup and Wets [35], see also [34, Example 9.35], says that a convex polyhedral map S is Lipschitz continuous on its domain, i.e., there exist $\kappa > 0$ such that

$$S(y') \subset S(y) + \kappa\|y'-y\|\mathscr{B} \quad \forall y,y' \in \operatorname{dom} S.$$

Since this is obviously stronger even than inner calmness at any $\bar{y}$, we obtain the following remarkable result, taking also into account the automatic closedness of the domain of convex polyhedral maps. For the sake of completeness, we include also the well-known result regarding the (outer) calmness due to Robinson [32], who used the name *upper Lipschitzness*.

Theorem 3.6. *Let $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a polyhedral set-valued map. Then there exists a constant κ such that S is inner calm* with constant κ wrt* $\operatorname{dom} S$ *at every $\bar{y} \in \mathbb{R}^m$ as well as calm with constant κ at every $\bar{y} \in \operatorname{dom} S$, i.e.,*

$$S(y) \subset S(\bar{y}) + \kappa \|y - \bar{y}\| \mathscr{B}$$

for all y near $\bar{y}$.

Note that, in inner calmness* we have found a suitable inner Lipschitzian property that, from the Lipschitzness of convex polyhedral maps, extends to the polyhedral maps. The key reason is that the inner calmness* is based on the suitable notion of inner semicompactness, which is preserved under finite unions. To the best of our knowledge, there has not been any inner Lipschitzian property known to hold for the polyhedral maps.

It seems quite intriguing that a map which is not Lipschitz continuous, just like the polyhedral maps in general, may still be inner calm* as well as (outer) calm. Interestingly, in [16, Theorem 8] Gfrerer and Outrata also established some sufficient conditions to ensure both, calmness and inner calmness, of certain class of solution maps. Perhaps a reasonable analysis can be achieved in some situations, where Lipschitness is typically assumed, by relaxing the assumption to fulfilment of some kind of the "two-sided" calmness.

3.2 Multiplier mappings

In [16, Proposition 4.1.], Gfrerer and Outrata showed the following remarkable result.

Proposition 3.7. *[16, Proposition 4.1] Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping having locally closed graph around $(x, \bar{y}) \in \operatorname{gph} M$ and assume that M is metrically subregular at $(x, \bar{y})$ with modulus κ. Then*

$$N_{M^{-1}(\bar{y})}(x) \subset \left\{ x^* \mid \exists y^* \in \kappa \|x^*\| \mathscr{B} : (x^*, y^*) \in N_{\operatorname{gph} M}(x, \bar{y}) \right\}. \tag{3.6}$$

In fact, [16, Proposition 4.1.] contains, apart from the additional result for tangent cone, the stronger, directional, estimate. Nevertheless, it can be easily seen that the above statement is really the essential one, while the directional version is just one of its admirable corollaries. Indeed, taking into account the straightforward fact that, given a direction $u \in \mathbb{R}^n$ as well as sequences $t_k \downarrow 0$ and $u_k \to u$, the metric subregularity of M in direction u at $(x, \bar{y})$ yields metric subregularity of M at every point $(x + t_k u_k, \bar{y}) \in \operatorname{gph} M$ with the same modulus, applying (3.6) at $(x + t_k u_k, \bar{y})$ and taking the limit, one immediately obtains

$$N_{M^{-1}(\bar{y})}(x; u) \subset \left\{ x^* \mid \exists y^* \in \kappa \|x^*\| \mathscr{B} : \ (x^*, y^*) \in N_{\operatorname{gph} M}((x, \bar{y}); (u, 0)) \right\}.$$

We point out that taking the corresponding limits is only possible thanks to the bound $\|y^*\| \le \kappa \|x^*\|$ in (3.6). More generally, in the area of second-order analysis and related stability and sensitivity issues, one often deals with a sequence $(x_k, x_k^*) \in \operatorname{gph} N_{M^{-1}(\bar{y})}$, where M is typically a constraint mapping, see below. Under the metric subregularity, one gets the existence of some suitable multipliers y_k^*, but needs to find also some limit multiplier y^* with $y_k^* \to y^*$. Hence, many authors used to assume boundedness of multipliers, for which they had to impose some stronger conditions, such as GMFCQ, see also [12] for more subtle approach. Thanks to the bound $\|y^*\| \le \kappa \|x^*\|$, however, we know that the metric subregularity alone is sufficient for this task. This fact was already heavily utilized in several recent works, see, e.g., [1, 3, 13, 14]. We also point out that the idea that the metric subregularity yields the bound $\|y^*\| \le \kappa \|x^*\|$ appeared already in the proof of [20, Theorem 4.1.]. Moreover, similar arguments were also used in [5, Lemma 3.2] in the setting of nonlinear programs.

For the constraint mapping $M(x) = \varphi(x) - Q$, where $Q \subset \mathbb{R}^m$ is a closed set and $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is calm at $x \in M^{-1}(0)$, one can easily show that $D^*M(x,0)(-y^*) \subset D^*\varphi(x)(y^*)$ if $y^* \in N_Q(\varphi(x))$, or, even the directional version

$$D^*M((x,0);(u,0))(-y^*) \subset \bigcup_{w \in D\varphi(x)(u):y^* \in N_Q(\varphi(x);w)} D^*\varphi(x;(u,w))(y^*).$$

valid under the calmness of φ at x in direction u, c.f. [2, Lemma 6.1]. Consequently, Proposition 3.7 yields also the standard as well as the directional pre-image calculus rule, c.f. [2, Theorem 3.1].

Corollary 3.8. *Assume that φ is calm at x in some direction $u \in \mathbb{R}^n$ and that the set-valued mapping $M(x) \rightrightarrows Q - \varphi(x)$ is metrically subregular at $(x,0)$ in direction u with modulus κ. Then, for $C := M^{-1}(0) = \varphi^{-1}(Q)$ we have*

$$N_C(x;u) \subset \{x^* \mid \exists v \in D\varphi(x)(u) \cap T_Q(\varphi(x)), \exists y^* \in N_Q(\varphi(x);v) \cap \kappa\|x^*\|\mathscr{B} : x^* \in D^*\varphi(x;(u,v))(y^*)\}.$$

Let us now discuss the result of Proposition 3.7 in the light of the inner semicompactness. To this end, suppose that we are not interested in finding an estimate for the whole set $N_{M^{-1}(\bar{y})}(x)$, but, instead, we have a fixed $x^* \in N_{M^{-1}(\bar{y})}(x)$ and we are looking for $y^* \in D^*M^{-1}(\bar{y},x)(-x^*)$. Clearly, the metric subregularity of M does not take into account the specific x^*.

On the other hand, assuming that the mapping

$$\Lambda(x,x^*) := D^*M^{-1}(\bar{y},x)(-x^*)$$

is inner semicompact at some point (x,x^*) wrt $\operatorname{gph} N_{M^{-1}(\bar{y})}$ means that, for every $(x_k,x_k^*) \to (x,x^*)$ with $(x_k,x_k^*) \in \operatorname{gph} N_{M^{-1}(\bar{y})}$, there exists $y_k^* \in D^*M^{-1}(\bar{y},x_k)(-x_k^*)$ together with $y^* \in D^*M^{-1}(\bar{y},x)(-x^*)$ such that $y_k^* \to y^*$. Of course, one loses the bound of $\|y^*\|$ here. Nevertheless, taking into account the discussion after Proposition 3.7, the inner semicompactness of Λ itself is often precisely of the main interest. This will become even more clear in the next sections.

Applying Proposition 3.7 at $(x_k,x_k^*) \in \operatorname{gph} N_{M^{-1}(\bar{y})}$ readily yields the following corollary.

Corollary 3.9. *Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping having locally closed graph around $(x,\bar{y}) \in \operatorname{gph} M$. Then the metric subregularity of M at $(x,\bar{y})$ implies the inner semicompactness of Λ at (x,x^*) wrt $\operatorname{gph} N_{M^{-1}(\bar{y})}$ for every $x^* \in N_{M^{-1}(\bar{y})}(x)$.*

We conclude this section by showing even stronger statement in case of the smooth constraints. Hence, assume that $M = \varphi(x) - Q$ for a closed set Q and a twice continuously differentiable mapping φ and set $C := M^{-1}(0) = \{x \mid \varphi(x) \in Q\}$. Denoting $\beta(x) := \nabla\varphi(x)$, the (multiplier) mapping Λ attains the standard form

$$\Lambda(x,x^*) := \{\lambda \in N_Q(\varphi(x)) \mid \beta(x)^T\lambda = x^*\},$$

where we are using λ instead of y^* to denote the multipliers. Note also that the metric subregularity of $M = \varphi(x) - Q$ at $(x,0)$ yields that locally around (x,x^*) one has $\operatorname{gph} N_C \subset \operatorname{dom}\Lambda$ and the equality holds, e.g., if Q is Clarke regular near $\varphi(x)$, in particular convex.

Theorem 3.10. *Let $(x,x^*) \in \operatorname{gph} N_C$ for $C = \varphi^{-1}(Q)$ and assume that the constraint mapping $M(x) = \varphi(x) - Q$ is metrically subregular at $(x,0)$. Then*

(i) the multiplier map Λ is inner semicompact at (x,x^*) wrt $\operatorname{gph} N_C$;

(ii) if Q is convex polyhedral, then Λ is even inner calm at (x,x^*) wrt its domain in the fuzzy sense.*

Proof. The first statement follows from Corollary 3.9.

To show the second one, assume that Q is convex polyhedral and consider a sequence $(x_k,x_k^*)\to(x,x^*)$ with $(x_k,x_k^*)\in\operatorname{dom}\Lambda$ and denote $(u_k,u_k^*):=(x_k-x,x_k^*-x^*)/t_k\to(u,u^*)\in\mathscr{S}$ for $t_k:=\|(x_k,x_k^*)-(x,x^*)\|$. The convexity of Q yields that $(x_k,x_k^*)\in\operatorname{gph}N_C$ and hence the inner semicompactness of Λ wrt $\operatorname{gph}N_C$ yields the existence of $\tilde\lambda_k\in N_Q(\varphi(x+t_ku_k))$ with

$$x^*+t_ku_k^*=\beta(x+t_ku_k)^T\tilde\lambda_k \tag{3.7}$$

as well as $\lambda\in N_Q(\varphi(x))$ with $x^*=\beta(x)^T\lambda$ such that $\tilde\lambda_k\to\lambda$.

The polyhedrality of Q yields the polyhedrality of $\operatorname{gph}N_Q$ and hence

$$\left(\frac{\varphi(x+t_ku_k)-\varphi(x)}{t_k},\frac{\tilde\lambda_k-\lambda}{t_k}\right)\in T_{\operatorname{gph}N_Q}(\varphi(x),\lambda)=\operatorname{gph}N_{\mathscr{K}_Q(\varphi(x),\lambda)},$$

by the reduction lemma formula (2.1). This means that

$$\begin{aligned}&(\varphi(x+t_ku_k)-\varphi(x))/t_k\in\mathscr{K}_Q(\varphi(x),\lambda),\ (\tilde\lambda_k-\lambda)/t_k\in\big(\mathscr{K}_Q(\varphi(x),\lambda)\big)^\circ,\\&\langle\varphi(x+t_ku_k)-\varphi(x),(\tilde\lambda_k-\lambda)/t_k\rangle=0.\end{aligned}$$

Hence, for every k there exists a face $\mathscr{F}_k:=\mathscr{K}_Q(\varphi(x),\lambda)\cap[(\tilde\lambda_k-\lambda)/t_k]^\perp$ of the critical cone with $\varphi(x+t_ku_k)-\varphi(x)\in\mathscr{F}_k$ and, due to the finiteness of faces of a polyhedral cone, we may assume that $\mathscr{F}_k\equiv\mathscr{F}$ with

$$(\tilde\lambda_k-\lambda)/t_k\in(\mathscr{K}_Q(\varphi(x),\lambda))^\circ\cap(\operatorname{sp}\mathscr{F})^\perp.$$

Moreover, the Taylor expansion of (3.7) yields

$$\beta(x)^T\frac{\tilde\lambda_k-\lambda}{t_k}=u_k^*-\nabla\langle\lambda,\beta\rangle(x)u+o(1). \tag{3.8}$$

Hence, we can now invoke Hoffman's lemma [4, Theorem 2.200] to find for every k some $\eta_k\in(\mathscr{K}_Q(\varphi(x),\lambda))^\circ\cap(\operatorname{sp}\mathscr{F})^\perp$ satisfying

$$\beta(x)^T\eta_k=\beta(x)^T\frac{\tilde\lambda_k-\lambda}{t_k}$$

and $\|\eta_k\|\le\alpha\|\beta(x)^T(\tilde\lambda_k-\lambda)/t_k\|$ for some constant $\alpha>0$ not depending on k. Since the right hand side of (3.8) is bounded, so is η_k and by possibly passing to a subsequence we can assume that η_k converges to some $\eta\in(\mathscr{K}_Q(\varphi(x),\lambda))^\circ\cap(\operatorname{sp}\mathscr{F})^\perp$ satisfying

$$\beta(x)^T\eta=u^*-\nabla\langle\lambda,\beta\rangle(x)u.$$

We have

$$\eta\in(\mathscr{K}_Q(\varphi(x),\lambda))^\circ=(T_Q(\varphi(x))\cap[\lambda]^\perp)^\circ=T_{N_Q(\varphi(x))}(0)+[\lambda]=T_{N_Q(\varphi(x))}(\lambda)$$

and the polyhedrality of $N_Q(\varphi(x))$ yields that $\lambda_k:=\lambda+t_k\eta\in N_Q(\varphi(x))$. Moreover, from $\eta\in(\operatorname{sp}\mathscr{F})^\perp$, $\varphi(x+t_ku_k)-\varphi(x)\in\mathscr{F}$ as well as $\langle\lambda,\varphi(x+t_ku_k)-\varphi(x)\rangle=0$ we infer

$$\langle\lambda_k,\varphi(x+t_ku_k)-\varphi(x)\rangle=0$$

and, consequently, $\lambda_k\in N_Q(\varphi(x+t_ku_k))$. Note that for $\kappa:=\|\eta\|$ we obtain $\|\lambda_k-\lambda\|=t_k\kappa$.

Finally, denoting

$$\tilde{x}_k^* := \beta(x+t_k u_k)^T \lambda_k = x^* + t_k u^* + o(t_k) = x_k^* + o(t_k)$$

yields the statement, taking also into account that the fact that locally $\operatorname{dom}\Lambda = \operatorname{gph} N_C$ holds due to the convexity of Q. $\square$

Remark 3.11. *Let us briefly comment the above results.*

1. *We did not use the fact that $\beta(x) := \nabla\varphi(x)$ in the proof. In other words, for arbitrary sufficiently smooth function $\beta(x)$ it holds that $\Lambda(x,x^*)$ is inner calm* at (x,x^*) wrt its domain in the fuzzy sense, provided Λ is inner semicompact at (x,x^*) wrt $\operatorname{gph} N_C$ and Q is convex polyhedral. This enables us to handle the parametrized setting in Section 5 with ease.*

2. *As already mentioned, in order to replace the inner semicompactness in (i) by the local boundedness, one has to impose a stronger assumption, such as GMFCQ. Moreover, in order to obtain the analogous results in terms of the inner semicontinuity and the inner calmness, one can, e.g., assume the uniqueness of the multiplier. In turn, one needs to strengthen the assumption even more and require some kind of the nondegeneracy.*

3. *Finally, note that our proof closely follows the arguments from the proof of [3, Theorem 5.3].*

4 Selected calculus rules in primal and dual form

This section contains the "forward" calculus rules based on the four inner-type conditions. To shorten and unburden the presentation, we put here the list of the assumptions that serves as a template. In the following, we will impose these assumptions on different mappings, hence the subscript.

Assumption 4.1. *Given a set-valued map $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a point $\bar{y} \in \mathbb{R}^m$, we assign the four labels to the following assumptions:*

(i_S) S is inner semicompact at $\bar{y}$ wrt its domain in direction v;

(ii_S) S is inner semicontinuous at $(\bar{y},\bar{x})$ wrt its domain in direction v for some $\bar{x} \in S(\bar{y})$;

(iii_S) S is inner calm at $\bar{y}$ wrt its domain in direction v;*

(iv_S) S is inner calm at $(\bar{y},\bar{x})$ wrt its domain in direction v for some $\bar{x} \in S(\bar{y})$.

4.1 Tangents and directional normals to image sets

In this section, we deal with the rules for image sets, which provide the base for all of the remaining calculus. Given a closed set $C \subset \mathbb{R}^n$ and a continuous mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^l$, set $Q := \varphi(C)$ and consider $\bar{y} \in Q$. Moreover, let $\Psi : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ be given by $\Psi(y) := \varphi^{-1}(y) \cap C$ and note that $\operatorname{dom}\Psi = Q$.

Theorem 4.2 (Tangents to image sets)**.** *Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}^l$ is calm at some $\bar{x} \in \Psi(\bar{y})$. Then, we have*

$$T_C(\bar{x}) \subset \{u \mid \exists v \in T_Q(\bar{y}) \text{ with } v \in D\varphi(\bar{x})u\}.$$

On the other hand, if (iii_Ψ) holds, then

$$T_Q(\bar{y}) \subset \bigcup_{\bar{x}\in\Psi(\bar{y})} \{v \mid \exists u \in T_C(\bar{x}) \text{ with } v \in D\varphi(\bar{x})u\}$$

and, moreover, if there exists $\bar{x} \in \Psi(\bar{y})$ such that (iv$_\Psi$) holds, then the estimate holds with this $\bar{x}$, i.e., the union over $\Psi(\bar{y})$ is superfluous.

Proof. Pick $u \in T_C(\bar{x})$. We will show a slightly stronger statement, namely that $D\varphi(\bar{x})u \cap T_Q(\bar{y}) \neq \emptyset$, provided φ is calm at $\bar{x}$ *in direction u*. Indeed, consider $t_k \downarrow 0$ and $u_k \to u$ with $\bar{x}+t_k u_k \in C$ and observe that

$$Q \ni \varphi(\bar{x}+t_k u_k) = \bar{y}+t_k v_k, \tag{4.9}$$

where $v_k := (\varphi(\bar{x}+t_k u_k) - \bar{y})/t_k$ is bounded by the assumed calmness in direction u and we may assume that $v_k \to v$ for some $v \in T_Q(\bar{y})$. Moreover, (4.9) can be written as $(\bar{x}+t_k u_k, \bar{y}+t_k v_k) \in \operatorname{gph}\varphi$, showing also $v \in D\varphi(\bar{x})u$.

Consider now $v \in T_Q(\bar{y})$ as well as $t_k \downarrow 0$ and $v_k \to v$ with $\bar{y}+t_k v_k \in Q$. As before, the inner calmness* of Ψ at $\bar{y}$ wrt its domain *in direction v* yields the existence of $x_k \in C$ and $\bar{x}$ with

$$\bar{y}+t_k v_k = \varphi(x_k) \ \text{ and } \ \|x_k - \bar{x}\| \leq \kappa t_k \|v_k\|$$

for some $\kappa > 0$. This means, however, that $(\bar{x}+t_k u_k, \bar{y}+t_k v_k) \in \operatorname{gph}\varphi$ for the bounded sequence $u_k := (x_k - \bar{x})/t_k$ and the existence of $u \in T_C(\bar{x})$ with $v \in D\varphi(\bar{x})u$ follows. Moreover, the closedness of C and the continuity of φ imply $\bar{x} \in \Psi(\bar{y})$.

The last statement now follows easily from the definition of inner calmness. □

Remark 4.3. *In order to keep things simpler, we assumed the standard inner calmness* and we will continue in this manner. Nevertheless, the statement holds also under the inner calmness** in the fuzzy sense, *since this yields $\tilde{x}_k \in C$ with*

$$\bar{y}+t_k v_k + o(t_k) = \varphi(\tilde{x}_k) \ \textit{ and } \ \|\tilde{x}_k - \bar{x}\| \leq \kappa t_k \|v_k\|$$

and the rest of the proof remains valid with $\tilde{v}_k := (\varphi(\tilde{x}_k) - \bar{y})/t_k \to v$.

The following lemma highlights the underlying principle behind the above theorem.

Lemma 4.4. *If $(\bar{y},\bar{x}) \in \operatorname{gph}\Psi$ and $u \in D\Psi(\bar{y},\bar{x})(v)$, then*

$$u \in T_C(\bar{x}), \quad v \in T_Q(\bar{y}), \ \textit{ and } \ v \in D\varphi(\bar{x})u.$$

Proof. Having $t_k \downarrow 0$ and $(u_k, v_k) \to (u,v)$ with $(\bar{y},\bar{x}) + t_k(v_k,u_k) \in \operatorname{gph}\Psi$ means precisely that $\bar{x}+t_k u_k \in C$, $\bar{y}+t_k v_k \in Q$, and $(\bar{x}+t_k u_k, \bar{y}+t_k v_k) \in \operatorname{gph}\varphi$. □

Remark 4.5. *Note that the proof of Theorem 4.2 in fact goes as follows. If $u \in T_C(\bar{x})$ and φ is calm at $\bar{x}$ in direction u, then there exists v with $u \in D\Psi(\bar{y},\bar{x})(v)$. On the other hand, if $v \in T_Q(\bar{y})$ and Ψ is inner calm* at $\bar{y}$ wrt its domain in direction v, then there exist $\bar{x} \in \Psi(\bar{y})$ and $u \in D\Psi(\bar{y},\bar{x})(v)$.*

In the differentiable case, $D\varphi(\bar{x})u = \{\nabla\varphi(\bar{x})u\}$ is a singleton and we obtain the following simpler estimates. Again, note that directional differentiability of φ is also sufficient.

Corollary 4.6 (Tangents to image sets - differentiable case)**.** *If φ is continuously differentiable, then*

$$T_Q(\bar{y}) \supset \bigcup_{\bar{x} \in \Psi(\bar{y})} \nabla\varphi(\bar{x}) T_C(\bar{x}).$$

The above inclusion becomes equality under (iii$_\Psi$), and, moreover, the union over $\Psi(\bar{y})$ is superfluous if even (iv$_\Psi$) holds for some $\bar{x} \in \Psi(\bar{y})$.

Naturally, the estimates for directional normals to image sets from [2, Theorem 3.2], as well as the estimates for directional subdifferentials and coderivatives which are based on this result, can also be enriched by the inner calmness* assumption. Thus, we will now summarize the enhanced results, mostly following the structure and notation from [2]. Since the proofs are presented in [2] and the needed adjustments are usually minor, we only briefly describe the ideas and principles here.

The estimates in next theorem will differ based on which of the four assumptions we utilize. Given a direction $v \in \mathbb{R}^l$, let us also denote

$$\Sigma(v) := \{y^* \mid D^*\varphi(\bar{x};(u,v))(y^*) \cap N_C(\bar{x};u) \neq \emptyset\}. \tag{4.10}$$

Trivial modifications of the proof of [2, Theorem 3.2] to incorporate assumption (iii$_\Psi$) yield the following result.

Theorem 4.7 (Directional normals to image sets)**.** *Consider a direction $v \in \mathbb{R}^l$ and assume that φ is Lipschitz continuous near every $\bar{x} \in \Psi(\bar{y})$ in all directions $u \in D\Psi(\bar{y},\bar{x})(v)$ and $u \in D\Psi(\bar{y},\bar{x})(0) \cap \mathscr{S}$. Then*

- *under (i$_\Psi$) in direction v, one has*

$$N_Q(\bar{y};v) \subset \bigcup_{\bar{x}\in\Psi(\bar{y})} \Big(\bigcup_{u\in D\Psi(\bar{y},\bar{x})(v)} \Sigma(v) \ \cup \bigcup_{u\in D\Psi(\bar{y},\bar{x})(0)\cap\mathscr{S}} \Sigma(0)\Big);$$

- *under (ii$_\Psi$) in direction v with the prescribed $\bar{x} \in \Psi(\bar{y})$, one has*

$$N_Q(\bar{y};v) \subset \bigcup_{u\in D\Psi(\bar{y},\bar{x})(v)} \Sigma(v) \ \cup \bigcup_{u\in D\Psi(\bar{y},\bar{x})(0)\cap\mathscr{S}} \Sigma(0);$$

- *under (iii$_\Psi$) in v, one has*

$$N_Q(\bar{y};v) \subset \bigcup_{\bar{x}\in\Psi(\bar{y})} \ \bigcup_{u\in D\Psi(\bar{y},\bar{x})(v)} \Sigma(v);$$

- *finally, under (iv$_\Psi$) in v with the prescribed $\bar{x} \in \Psi(\bar{y})$, one has*

$$N_Q(\bar{y};v) \subset \bigcup_{u\in D\Psi(\bar{y},\bar{x})(v)} \Sigma(v) = \bigcup_{u\in D\Psi(\bar{y},\bar{x})(v)} \{y^* \mid D^*\varphi(\bar{x};(u,v))(y^*) \cap N_C(\bar{x};u) \neq \emptyset\}.$$

Naturally, with strengthening the assumptions of Ψ, one can relax the Lipschitzness of φ accordingly. E.g., under (iv$_\Psi$) one only needs φ to be Lipschitz near the fixed $\bar{x} \in \Psi(\bar{y})$ in direction $u \in D\Psi(\bar{y},\bar{x})(v)$.

Note that directions $u \in D\Psi(\bar{y},\bar{x})(v)$ and $u \in D\Psi(\bar{y},\bar{x})(0) \cap \mathscr{S}$ can be written in terms of the data C, φ and Q as clarified in Lemma 4.4.

4.2 Directional subdifferentials of value function

Following [2], we consider directional limiting subdifferentials with respect to a pair of directions. More precisely, given an extended real-valued function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, a point $\bar{x} \in \operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) \in \mathbb{R}\}$ in the domain of f as well as a direction $(u,\mu) \in \mathbb{R}^{n+1}$, the *directional limiting subdiferential* of f at $\bar{x}$ *in direction* (u,μ) is given via

$$\partial f(\bar{x};(u,\mu)) := \{\xi \in \mathbb{R}^n \mid (\xi,-1) \in N_{\operatorname{epi} f}((\bar{x},f(\bar{x}));(u,\mu))\}, \tag{4.11}$$

where $\operatorname{epi} f := \{(x,\alpha) \in \mathbb{R}^{n+1} \mid a \geq f(x)\}$ stands for the epigraph of f. Moreover, we say that f is lower semicontinuous (lsc) around $\bar{x}$ if $\operatorname{epi} f$ is locally closed around $(\bar{x}, f(\bar{x}))$.

In order to discuss the *value* (or *marginal*) function, consider an lsc function $f : \mathbb{R}^n \times \mathbb{R}^l \to \bar{\mathbb{R}}$, set $\vartheta(y) = \inf_{x \in \mathbb{R}^n} f(x,y)$ and assume that ϑ is finite at $\bar{y}$. Moreover, let $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ be the solution mapping given by $S(y) = \arg\min f(\cdot, y)$ and consider a direction $(v,\mu) \in \mathbb{R}^{l+1}$. Following the approach from [25, Proposition 3.2., Proposition 3.3.], we impose the crucial inner condition on mapping S, but *without* the restriction to the domain.

Under the mildest assumption (i_S), the connection between the subdifferentials of ϑ and f is provided by $\operatorname{epi} \vartheta = \varphi(\operatorname{epi} f)$ with $\varphi : \mathbb{R}^{n+l+1} \to \mathbb{R}^{l+1}$ given by $\varphi(x,y,\alpha) = (y,\alpha)$, see [34, Theorem 10.13]. Hence, in order to apply Theorem 4.7, consider also the map

$$\Psi(y,\alpha) := \varphi^{-1}(y,\alpha) \cap \operatorname{epi} f = \{(x,y,\alpha) \mid (x,y,\alpha) \in \operatorname{epi} f\}, \tag{4.12}$$

and note that $\operatorname{dom} \Psi = \operatorname{epi} \vartheta$.

Based on the simple observation that

$$\{(x,y,\alpha) \mid (y,\alpha) \in \operatorname{epi} \vartheta, x \in S(y)\} \subset \Psi(y,\alpha), \tag{4.13}$$

it was shown in [2, Proof of Theorem 4.2] that the assumptions (i_S), (ii_S) and (iv_S) imply the corresponding assumptions on Ψ. To give the reader a better insight, we prove that (iii_S) implies (iii_Ψ) to obtain the following lemma.

Lemma 4.8. *The assumptions 4.1 on S (without the restriction to the domain of S) imply the corresponding assumptions on Ψ, namely (i_S) implies (i_Ψ), (ii_S) implies (ii_Ψ), (iii_S) implies (iii_Ψ) and (iv_S) implies (iv_Ψ).*

Proof. Consider $\operatorname{epi} \vartheta \ni (y_k, \alpha_k) \to (\bar{y}, \vartheta(\bar{y}))$ from direction (v,μ), i.e., $y_k \to \bar{y}$ from direction v. The inner calmness* of S at $\bar{y}$ in direction v yields the existence of $\kappa > 0$ as well as $\bar{x}$ and a sequence $x_k \to \bar{x}$ such that, by passing to a subsequence, we have $x_k \in S(y_k)$ and $\|x_k - \bar{x}\| \leq \kappa \|y_k - \bar{y}\|$. Hence $(x_k, y_k, \alpha_k) \in \Psi(y_k, \alpha_k)$ by (4.13) with $(x_k, y_k, \alpha_k) \to (\bar{x}, \bar{y}, \vartheta(\bar{y}))$. Moreover,

$$\begin{aligned} \|(x_k,y_k,\alpha_k) - (\bar{x},\bar{y},\vartheta(\bar{y}))\| &\leq \|x_k - \bar{x}\| + \|(y_k,\alpha_k) - (\bar{y},\vartheta(\bar{y}))\| \\ &\leq \kappa\|y_k - \bar{y}\| + \|(y_k,\alpha_k) - (\bar{y},\vartheta(\bar{y}))\| \leq (\kappa+1)\|(y_k,\alpha_k) - (\bar{y},\vartheta(\bar{y}))\|, \end{aligned}$$

showing the inner calmness* of Ψ at $(\bar{y}, \vartheta(\bar{y}))$ wrt $\operatorname{dom} \Psi = \operatorname{epi} \vartheta$ in direction (v,μ). □

Naturally, one can also formulate the next theorem in terms of Ψ. We stress again that the assumption on S are understood without the restriction to the domain of S.

Theorem 4.9 (Directional subdifferentials of value function)**.** *Let $y^* \in \partial\vartheta(\bar{y}; (v,\mu))$. Then*

- *under (i_S) in direction v, one has*

$$(0,y^*) \in \bigcup_{\bar{x} \in S(\bar{y})} \Big(\bigcup_{u \in DS(\bar{y},\bar{x})(v)} \partial f((\bar{x},\bar{y});(u,v,\mu)) \cup \bigcup_{u \in DS(\bar{y},\bar{x})(0) \cap \mathscr{S}} \partial f((\bar{x},\bar{y});(u,0,0)) \Big);$$

- *under (ii_S) in direction v with the prescribed $\bar{x} \in S(\bar{y})$, one has*

$$(0,y^*) \in \bigcup_{u \in DS(\bar{y},\bar{x})(v)} \partial f((\bar{x},\bar{y});(u,v,\mu)) \cup \bigcup_{u \in DS(\bar{y},\bar{x})(0) \cap \mathscr{S}} \partial f((\bar{x},\bar{y});(u,0,0));$$

- *under (iii$_S$) in v, one has*

$$(0,y^*)\in \bigcup_{\bar{x}\in S(\bar{y})}\ \bigcup_{u\in DS(\bar{y},\bar{x})(v)} \partial f((\bar{x},\bar{y});(u,v,\mu));$$

- *under (iv$_S$) in v with the prescribed $\bar{x}\in S(\bar{y})$, one has*

$$(0,y^*)\in \bigcup_{u\in DS(\bar{y},\bar{x})(v)} \partial f((\bar{x},\bar{y});(u,v,\mu)).$$

The proof follows from Lemma 4.8 and Theorem 4.7, see also [2, Theorem 4.2] for more details. We point out that, in order to conclude that the estimates hold also with some $\bar{x}\in S(\bar{y})$ instead of $\bar{x}$ with $(\bar{x},\bar{y},\vartheta(\bar{y}))\in\Psi(\bar{y},\vartheta(\bar{y}))$, one actually has to look into the proof of [2, Theorem 3.2] and put it together with the arguments from the proof of Lemma 4.8. The same applies for picking the directions u from $DS(\bar{y},\bar{x})(v)$ and $DS(\bar{y},\bar{x})(0)\cap\mathscr{S}$, naturally. Alternatively, one can also prove the theorem directly, without relying on Theorem 4.7.

4.3 Chain rule and sum rule for set-valued maps, primal and dual form

Let us begin with the chain rule. To this end, consider the mappings $S_1:\mathbb{R}^n\rightrightarrows\mathbb{R}^m, S_2:\mathbb{R}^m\rightrightarrows\mathbb{R}^s$ with closed graphs, i.e., S_1 and S_2 are outer semicontinuous (osc), and set $S=S_2\circ S_1$. Let $(\bar{x},\bar{z})\in\operatorname{gph} S$ and $(u,w)\in\mathbb{R}^n\times\mathbb{R}^s$ be a pair of directions. Finally, consider the "intermediate" map $\Xi:\mathbb{R}^n\times\mathbb{R}^s\rightrightarrows\mathbb{R}^m$ defined by

$$\Xi(x,z):=S_1(x)\cap S_2^{-1}(z)=\{y\in S_1(x)\mid z\in S_2(y)\}.$$

Following the approach from [34, Theorem 10.37] one has that

$$\operatorname{dom}\Xi=\operatorname{gph} S=\varphi\big(\phi^{-1}(\operatorname{gph} S_1\times\operatorname{gph} S_2)\big),$$

where $\varphi:(x,y,z)\mapsto(x,z)$ and $\phi:(x,y,z)\mapsto(x,y,y,z)$. This suggests applying first the image set rule and then the pre-image set rule.

Note that

$$\begin{aligned}\Psi(x,z) &:= \varphi^{-1}(x,z)\cap\phi^{-1}(\operatorname{gph} S_1\times\operatorname{gph} S_2)=\{(x,y,z)\mid (x,y)\in\operatorname{gph} S_1,(y,z)\in\operatorname{gph} S_2)\}\\ &= \{(x,y,z)\mid y\in\Xi(x,z)\},\end{aligned}$$

so it is not surprising that the crucial assumptions 4.1 needed for the image set rule can be imposed of Ξ, instead of Ψ. Naturally, these conditions are understood wrt to point $(\bar{x},\bar{z})$, domain $\operatorname{dom}\Xi=\operatorname{gph} S$ and some $\bar{y}\in\Xi(\bar{x},\bar{z})$.

On the other hand, for the pre-image set rule we will need metric subregularity of the constraint mapping

$$F(x,y,z):=\operatorname{gph} S_1\times\operatorname{gph} S_2-\phi(x,y,z)=\begin{bmatrix}\operatorname{gph} S_1-(x,y)\\ \operatorname{gph} S_2-(y,z)\end{bmatrix}.$$

We first propose the following *chain rule for graphical derivatives*.

Theorem 4.10 (Chain rule for graphical derivatives)**.** *Under (iii$_\Xi$) it holds that*

$$DS(\bar{x},\bar{z})\subset\bigcup_{\bar{y}\in\Xi(\bar{x},\bar{z})} DS_2(\bar{y},\bar{z})\circ DS_1(\bar{x},\bar{y}) \tag{4.14}$$

and the union over $\Xi(\bar{x},\bar{z})$ becomes superfluous if (iv$_\Xi$) holds with the prescribed $\bar{y}\in\Xi(\bar{x},\bar{z})$. On the other hand, given some $\bar{y}\in\Xi(\bar{x},\bar{z})$,

$$DS_2(\bar{y},\bar{z})\circ DS_1(\bar{x},\bar{y})\subset DS(\bar{x},\bar{z}) \tag{4.15}$$

holds provided F is metrically subregular at $((\bar{x},\bar{y},\bar{z}),(0,0))$ and

$$(u,v,v,w)\in T_{\operatorname{gph} S_1\times\operatorname{gph} S_2}(\bar{x},\bar{y},\bar{y},\bar{z}) \iff (u,v)\in T_{\operatorname{gph} S_1}(\bar{x},\bar{y}),\,(v,w)\in T_{\operatorname{gph} S_2}(\bar{y},\bar{z}). \tag{4.16}$$

Proof. The proof follows from Corollaries 4.6 and 3.8, taking also into account that inclusion $\subset$ in formula (4.16) always holds due to the well-known fact that

$$T_{\operatorname{gph} S_1\times\operatorname{gph} S_2}(\bar{x},\bar{y},\bar{y},\bar{z})\subset T_{\operatorname{gph} S_1}(\bar{x},\bar{y})\times T_{\operatorname{gph} S_1}(\bar{y},\bar{z}),$$

see [34, Proposition 6.41]. $\square$

For more details we refer to the proof of [2, Theorem 5.1]. Let us take a look into the underlying principle in terms of the graphical derivative of Ξ. Realizing that

$$\operatorname{gph} S=\varphi(\operatorname{gph}\Xi),\quad \operatorname{gph}\Xi=\phi^{-1}(\operatorname{gph} S_1\times\operatorname{gph} S_2)$$

and taking into account [34, Proposition 6.41] immediately gives the following lemma.

Lemma 4.11. *Let $\bar{y}\in\Xi(\bar{x},\bar{z})$ and $v\in D\Xi((\bar{x},\bar{z}),\bar{y})(u,w)$. Then*

$$w\in DS(\bar{x},\bar{z})(u)\ \text{ and }\ w\in DS_2(\bar{y},\bar{z})\circ DS_1(\bar{x},\bar{y})(u).$$

Remark 4.12. *In Theorem 4.10, assumption (iii$_\Xi$) (or (iv$_\Xi$)) in fact allows one to estimate $DS(\bar{x},\bar{z})$ via $D\Xi((\bar{x},\bar{z}),\bar{y})$, while the metric subregularity of F, together with* (4.16)*, provide the estimate of $DS_2(\bar{y},\bar{z})\circ DS_1(\bar{x},\bar{y})$ via $D\Xi((\bar{x},\bar{z}),\bar{y})$. Naturally, these two assumptions can be replaced by asking directly for the implication*

$$w\in DS_2(\bar{y},\bar{z})\circ DS_1(\bar{x},\bar{y})(u) \implies \exists\, v\in D\Xi((\bar{x},\bar{z}),\bar{y})(u,w). \tag{4.17}$$

Note that in [34, p. 454], the authors argue that it is difficult to obtain reasonable chain rule for graphical derivatives, since the image set and pre-image set rules for tangent cones in general work in “opposite direction”, see [34, Theorems 6.31 and 6.43]. More precisely, the upper estimate of the tangent cone to image set in [34, Theorem 6.43] was obtained only under quite restrictive assumptions. This also emphasizes the importance of the inner calmness and the inner calmness*.

Theorem 4.13 (Directional coderivative chain rule)**.** *Assume that (i$_\Xi$) in direction (u,w) holds and that F is metrically subregular at $((\bar{x},\bar{y},\bar{z}),(0,0))$ for all $\bar{y}\in\Xi(\bar{x},\bar{z})$ in directions (u,v,w) with $v\in D\Xi((\bar{x},\bar{z}),\bar{y})(u,w)$ and in directions $(0,v,0)$ with $v\in D\Xi((\bar{x},\bar{z}),\bar{y})(0,0)\cap\mathscr{S}$. Then one has*

$$\begin{aligned} D^*S((\bar{x},\bar{z});(u,w))\subset \bigcup_{\bar{y}\in\Xi(\bar{x},\bar{z})}\Big(&\bigcup_{v\in D\Xi((\bar{x},\bar{z}),\bar{y})(u,w)} D^*S_1((\bar{x},\bar{y});(u,v))\circ D^*S_2((\bar{y},\bar{z});(v,w))\\ &\cup\bigcup_{v\in D\Xi((\bar{x},\bar{z}),\bar{y})(0,0)\cap\mathscr{S}} D^*S_1((\bar{x},\bar{y});(0,v))\circ D^*S_2((\bar{y},\bar{z});(v,0))\Big). \end{aligned} \tag{4.18}$$

Moreover, assuming the stronger properties (ii$_\Xi$), (iii$_\Xi$) or (iv$_\Xi$) in direction (u,w) yields the evident simplifications in the above estimate, whereby the assumption of metric subregularity of F can be accordingly restricted to the corresponding points and directions, analogously to Theorem 4.7.

Next we briefly discuss the situation when one of the mappings is single-valued. We begin by the following lemma.

Lemma 4.14. *Let $\bar{y} \in \Xi(\bar{x},\bar{z})$. Then* (4.17) *holds provided one of the following conditions is satisfied:*

(a) S_1 is single-valued, differentiable at $\bar{x}$ with $\nabla S_1(\bar{x})$ being injective and S_1^{-1} is inner calm at $(\bar{y},\bar{x})$. This holds, in particular, if S_1 is linear and injective.

(b) S_2 is single-valued and differentiable at $\bar{y}$.

Proof. Consider (u,v,w) with $(u,v) \in T_{\operatorname{gph} S_1}(\bar{x},\bar{y})$ and $(v,w) \in T_{\operatorname{gph} S_2}(\bar{y},\bar{z})$ and let $t_k \downarrow 0$, $(u_k,v_k) \to (u,v)$ and $\tau_k \downarrow 0$, $(\tilde{v}_k,w_k) \to (v,w)$ with

$$\bar{y}+t_k v_k \in S_1(\bar{x}+t_k u_k) \text{ and } \bar{z}+\tau_k w_k \in S_2(\bar{y}+\tau_k \tilde{v}_k).$$

Under (a), we obtain from the inner calmness of S_1^{-1} the existence of $\kappa > 0$ and x_k with $\|x_k-\bar{x}\|/\tau_k \le \kappa\|\tilde{v}_k\|$ and

$$\bar{y}+\tau_k\tilde{v}_k = S_1(x_k) = \bar{y}+\nabla S_1(\bar{x})(x_k-\bar{x})+o(\|x_k-\bar{x}\|).$$

Hence, by passing to a subsequence we may assume that $\tilde{u}_k := (x_k-\bar{x})/\tau_k \to \tilde{u}$ and we conclude that $v = \nabla S_1(\bar{x})\tilde{u}$, showing that $\tilde{u} = u$ by the assumed injectivity of $\nabla S_1(\bar{x})$. This means, however, that $(\bar{x},\bar{z},\bar{y})+\tau_k(\tilde{u}_k,w_k,\tilde{v}_k) \in \operatorname{gph}\Xi$ and (4.17) follows.

On the other hand, (b) yields $S_2(\bar{y}+t_k v_k) = S_2(\bar{y})+t_k\hat{w}_k$ for $\hat{w}_k = \nabla S_2(\bar{y})v+o(1) \to w$. Thus $(\bar{x},\bar{z},\bar{y})+t_k(u_k,\hat{w}_k,v_k) \in \operatorname{gph}\Xi$ and (4.17) follows again. □

It is easy to see that if S_1 is single-valued and calm at $\bar{x}$, then Ξ is single- (or empty-) valued and the strongest assumption (iv$_\Xi$) is fulfilled. Hence, we obtain the following corollary.

Corollary 4.15. *Assume that S_1 is single-valued and calm at $\bar{x}$. Then*

$$DS(\bar{x},\bar{z}) \subset DS_2(S_1(\bar{x}),\bar{z}) \circ DS_1(\bar{x})$$

and the above inclusion becomes equality under (4.17)*. Moreover,* (4.17) *holds, in particular, if either F is metrically subregular at $((\bar{x},\bar{y},\bar{z}),(0,0))$ and* (4.16) *holds, or, alternatively, if condition (a) of Lemma 4.14 is satisfied.*

For the corresponding rule for coderivatives, we refer to [2, Corollary 5.1], noting that the Lipschitzness assumption on S_1 can be relaxed to calmness.

Now if S_2 is single-valued, on the basis of Lemma 4.14, we infer the following result.

Corollary 4.16. *Assume that S_2 is single-valued. Then (iii$_\Xi$) yields* (4.14) *and the union becomes superfluous under (iv$_\Xi$). On the other hand, given $\bar{y} \in \Xi(\bar{x},\bar{z})$,* (4.15) *holds under* (4.17)*, which follows, in particular, if S_2 is also differentiable at $\bar{y}$.*

Again, we skip the corresponding result for coderivatives, since it does not bring any new insights. Indeed, apart from the evident simplifications in estimate (4.18), one just does not have to impose the subregularity assumption, provided S_2 is Lipschitz continuous, see also [2, Corollary 5.2].

Next, we present the sum rule, where for simplicity we consider only the sum of two maps, i.e., $S = S_1+S_2$ for osc mappings $S_1,S_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Further, let $(\bar{x},\bar{z}) \in \operatorname{gph} S$ and $(u,w) \in \mathbb{R}^n \times \mathbb{R}^m$ be a pair of directions.

Here, the computation is done by applying the chain rule twice, since

$$S = F_2 \circ \mathscr{P} \text{ for } \mathscr{P} : x \rightrightarrows S_1(x) \times S_2(x) \text{ and } F_2 : (y_1,y_2) \to y_1+y_2,$$

taking into account that $\mathscr{P}$ can be again written as a composition $S_o \circ F_1$ for $F_1 : x \to (x,x)$ and $S_o : (q_1, q_2) \rightrightarrows S_1(q_1) \times S_2(q_2)$.

We point out here that the mapping $\mathscr{P}$ is interesting on its own, since it naturally apears whenever one is dealing with a mapping given by some operation applied to two mappings. In another words, $\mathscr{P}$ can be composed with other mappings to obtain interesting results. Hence, we first compute the generalized derivatives of $\mathscr{P}$. To this end, we introduce the following constraint mapping

$$F(x,q,y) := \begin{bmatrix} (x,x) - (q_1,q_2) \\ \operatorname{gph} S_1 - (q_1,y_1) \\ \operatorname{gph} S_2 - (q_2,y_2) \end{bmatrix}$$

and let $(\bar{x},\bar{y}) = (\bar{x},(\bar{y}_1,\bar{y}_1)) \in \operatorname{gph}\mathscr{P}$ and $(u,v) = (u,(v_1,v_2)) \in \mathbb{R}^n \times \left(\mathbb{R}^m \times \mathbb{R}^m\right)$ be a pair of directions. We use here the simplified map F with the first row replaced by $F_1(x) - (q_1,q_2)$. It is not difficult to show that this is possible due to the Lipschitzness of F_1.

Since $F_1 : x \to (x,x)$ clearly fulfills condition (a) of Lemma 4.14 and

$$\operatorname{gph} S_0 = \sigma(\operatorname{gph} S_1 \times \operatorname{gph} S_2),$$

where σ merely permutes the variables, we readily obtain the following result.

Proposition 4.17. *We have*

$$D\mathscr{P}(\bar{x},\bar{y}) \subset \big(DS_1(\bar{x},\bar{y}_1), DS_1(\bar{x},\bar{y}_2)\big)$$

and for equality one needs to assume

$$T_{\operatorname{gph} S_1 \times \operatorname{gph} S_2}(\bar{x},\bar{y}_1,\bar{x},\bar{y}_2) = T_{\operatorname{gph} S_1}(\bar{x},\bar{y}_1) \times T_{\operatorname{gph} S_2}(\bar{x},\bar{y}_2). \tag{4.19}$$

In addition, if F is metrically subregular at $\big((\bar{x},\bar{x},\bar{x},\bar{y}_1,\bar{y}_2),(0,\ldots,0)\big)$ in direction (u,u,u,v_1,v_2), then

$$D^*\mathscr{P}((\bar{x},\bar{y});(u,v)) \subset \big(D^*S_1((\bar{x},\bar{y}_1);(u,v_1)), D^*S_1((\bar{x},\bar{y}_2);(u,v_2))\big). \tag{4.20}$$

Remark 4.18. *If one of the mappings S_1, S_2 is single-valued and differentiable at $\bar{x}$, one can easily see that* (4.19) *as well as the metric subregularity of F hold and we have* (4.20) *and $D\mathscr{P}(\bar{x},\bar{y}) = \big(DS_1(\bar{x},\bar{y}_1), DS_1(\bar{x},\bar{y}_2)\big)$.*

Now we may proceed with the sum rule. In order to deal with the outer composition $F_2 \circ \mathscr{P}$ we will need the corresponding "intermediate" mapping. Thus, consider the map $\Xi : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m \times \mathbb{R}^m$ defined by

$$\Xi(x,z) := \mathscr{P}(x) \cap F_2^{-1}(z) = \{y = (y_1,y_2) \in \mathbb{R}^m \times \mathbb{R}^m \mid y_1 \in S_1(x), y_2 \in S_2(x), y_1 + y_2 = z\}.$$

Theorem 4.19 (Sum rule for graphical derivatives)**.** *Under (iii$_\Xi$) it holds that*

$$DS(\bar{x},\bar{z}) \subset \bigcup_{\bar{y} \in \Xi(\bar{x},\bar{z})} DS_1(\bar{x},\bar{y}_1) + DS_2(\bar{x},\bar{y}_2) \tag{4.21}$$

and the union over $\Xi(\bar{x},\bar{z})$ becomes superfluous if (iv$_\Xi$) holds with the prescribed $\bar{y} \in \Xi(\bar{x},\bar{z})$. On the other hand, given some $\bar{y} \in \Xi(\bar{x},\bar{z})$ such that (4.19) *holds, we have*

$$DS_1(\bar{x},\bar{y}_1) + DS_2(\bar{x},\bar{y}_2) \subset DS(\bar{x},\bar{z}). \tag{4.22}$$

In addition, the sum rule for the directional coderivatives can be formulated analogously to the chain rule from Theorem 4.13 with the main estimate (4.18) replaced by

$$D^*S((\bar{x},\bar{z});(u,w)) \subset \bigcup_{\bar{y}\in\Xi(\bar{x},\bar{z})} \Big(\bigcup_{v\in D\Xi((\bar{x},\bar{z}),\bar{y})(u,w)} D^*S_1((\bar{x},\bar{y}_1);(u,v_1)) + D^*S_2((\bar{x},\bar{y}_2);(u,v_2)) \\ \cup \bigcup_{v\in D\Xi((\bar{x},\bar{z}),\bar{y})(0,0)\cap\mathbb{S}} D^*S_1((\bar{x},\bar{y}_1);(0,v_1)) + D^*S_2((\bar{x},\bar{y}_2);(0,v_2)) \Big).$$

We also refer to [2, Theorem 5.2] for more details.

We conclude this section by a simple product rule, where $S = S_1 \cdot S_2$ for an osc mapping $S_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a single-valued differentiable $S_1 : \mathbb{R}^n \to (\mathbb{R}^m)^l$. More precisely, $S_1(x)$ is a matrix of m rows and l columns and so $S(x) = \bigcup_{y\in S_2(x)} S_1(x)^T y = \bigcup_{y\in S_2(x)} \langle y, S_1\rangle(x)$. Further, let $(\bar{x},\bar{z}) \in \operatorname{gph} S$ and $(u,w) \in \mathbb{R}^n \times \mathbb{R}^l$ be a pair of directions.

Clearly, we can make use of mapping $\mathscr{P}$ again, since S can be written as $F_3 \circ \mathscr{P}$ for $F_3 : (\mathbb{R}^m)^l \times \mathbb{R}^m \to \mathbb{R}^l$ given by $F(A,y) = A^T y$. Taking into account the single-valuedness of S_1, instead of $\mathscr{P}(x) \cap F_3^{-1}(z)$, we can use the following simplified “intermediate” mapping $\widetilde{\Xi} : \mathbb{R}^n \times \mathbb{R}^l \rightrightarrows \mathbb{R}^m$ given by

$$\widetilde{\Xi}(x,z) := \{y \in S_2(x) \mid S_1(x)^T y = z\}. \tag{4.23}$$

On the basis of Remark 4.18 and the simple structure of F_3, we arrive at the following elegant result.

Theorem 4.20 (Product rule). *Under (iii$_{\widetilde{\Xi}}$), for all $z^* \in \mathbb{R}^l$ it holds that*

$$\begin{aligned} DS(\bar{x},\bar{z})(u) &= \bigcup_{\bar{y}\in\widetilde{\Xi}(\bar{x},\bar{z})} \nabla\langle\bar{y},S_1\rangle(\bar{x})u + S_1(\bar{x})^T DS_2(\bar{x},\bar{y})(u), \\ D^*S((\bar{x},\bar{z});(u,w))(z^*) &\subset \bigcup_{\bar{y}\in\widetilde{\Xi}(\bar{x},\bar{z})} \bigcup_{v\in D\widetilde{\Xi}((\bar{x},\bar{z}),\bar{y})(u,w)} \nabla\langle\bar{y},S_1\rangle(\bar{x})^T z^* + D^*S_2((\bar{x},\bar{y});(u,v))(S_1(\bar{x})z^*). \end{aligned}$$

The union over $\widetilde{\Xi}(\bar{x},\bar{z})$ become superfluous if (iv$_{\widetilde{\Xi}}$) holds with the prescribed $\bar{y} \in \widetilde{\Xi}(\bar{x},\bar{z})$. Moreover, under just (i$_{\widetilde{\Xi}}$) we have

$$D^*S(\bar{x},\bar{z})(z^*) \subset \bigcup_{\bar{y}\in\widetilde{\Xi}(\bar{x},\bar{z})} \nabla\langle\bar{y},S_1\rangle(\bar{x})^T z^* + D^*S_2(\bar{x},\bar{y})(S_1(\bar{x})z^*).$$

5 Application: Generalized derivatives of the normal-cone mapping

In this section, we apply the proposed calculus rules to compute the graphical derivative of the normal-cone mapping and to estimate its directional limiting coderivative. On one hand, we will show that our calculus-based approach is very robust and easy to apply in the variety of settings. On the other hand, we will also see its shortcomings compared to the results obtained by a direct computation. In the first part, we deal with the simple constraints and afterwards we consider the parametrized setting.

As a specific application we derive very interesting results regarding the newly developed concept of *semismoothness**, which plays the crucial role in the new remarkable Newton method for generalized equations [18].

5.1 Simple constraints

Consider the simple constraint system $g(x) \in D$, where $D \subset \mathbb{R}^s$ is a convex polyhedral set and $g : \mathbb{R}^n \to \mathbb{R}^s$ is twice continuously differentiable and denote

$$\Gamma := g^{-1}(D) = \{x \in \mathbb{R}^n \mid g(x) \in D\}. \tag{5.24}$$

Moreover, fix $\bar{x} \in \Gamma$ and assume that the constraint map $g(x) - D$ is metrically subregular at $\bar{x}$, which, in turn, means that the metric subregularity holds at all x near to $\bar{x}$ and, consequently,

$$N_\Gamma(x) = \nabla g(x)^T N_D(g(x)).$$

Given also $x^* \in N_\Gamma(x)$, Theorem 3.10 yields that the multiplier mapping

$$\Lambda(x,x^*) = \{\lambda \in N_D(g(x)) \mid \nabla g(x)^T \lambda = x^*\}$$

is inner calm* at (x,x^*) wrt its domain (in the fuzzy sense). Noting that Λ is precisely the "intermediate" mapping $\widetilde{\Xi}$ from (4.23) appearing in the product rule from Theorem 4.20, we immediately obtain the following auxiliary estimates.

Proposition 5.1. *For all $u \in \mathbb{R}^n$, we have*

$$DN_\Gamma(x,x^*)(u) = \bigcup_{\lambda \in \Lambda(x,x^*)} \nabla^2 \langle \lambda, g \rangle(x)u + \nabla g(x)^T D(N_D \circ g)(x,\lambda)(u)$$

and for all $w^* \in \mathbb{R}^n$ and $u^* \in DN_\Gamma(x,x^*)(u)$ we obtain

$$\begin{aligned} &D^* N_\Gamma((x,x^*);(u,u^*))(w^*) \\ &\quad \subset \bigcup_{\lambda \in \Lambda(x,x^*)} \ \bigcup_{\eta \in D\Lambda((x,x^*),\lambda)(u,u^*)} \nabla^2 \langle \lambda, g \rangle(x) w^* + D^*(N_D \circ g)((x,\lambda);(u,\eta))(\nabla g(x) w^*). \end{aligned}$$

Note that the analogous estimate for the standard limiting coderivative holds for any closed set D, which is Clarke regular near $g(x)$, in particular convex.

These estimates, however, still contain a composition $N_D \circ g$ that needs to be dealt with. Applying Corollary 4.15 as well as the reduction lemma formula (2.1) yields the estimate

$$\begin{aligned} DN_\Gamma(x,x^*)(u) &\subset \bigcup_{\lambda \in \Lambda(x,x^*)} \nabla^2 \langle \lambda, g \rangle(x)u + \nabla g(x)^T DN_D(g(x),\lambda)(\nabla g(x)u) \\ &= \bigcup_{\lambda \in \Lambda(x,x^*)} \nabla^2 \langle \lambda, g \rangle(x)u + \nabla g(x)^T N_{\mathscr{K}_D(g(x),\lambda)}(\nabla g(x)u). \end{aligned}$$

In order to reverse the above inclusion, one seemingly needs to impose some constraint qualification, e.g., the metric subregularity of the mapping $(x,\lambda) \rightrightarrows (g(x),\lambda) - \operatorname{gph} N_D$. Note also that the same applies if one wants to proceed with the estimates for coderivatives.

Moreover, we can further adjust the above estimate. Indeed, for every $\lambda \in \Lambda(x,x^*)$ it holds that

$$\mathscr{K}_\Gamma(x,x^*) = \nabla g(x)^{-1} \mathscr{K}_D(g(x),\lambda).$$

Since $u \rightrightarrows \nabla g(x)u - \mathscr{K}_D(g(x),\lambda)$ is a polyhedral map and hence metrically subregular by the Robinson's results [32, Proposition 1], see also Theorem 3.6, we obtain

$$N_{\mathscr{K}_\Gamma(x,x^*)}(u) = \nabla g(x)^T N_{\mathscr{K}_D(g(x),\lambda)}(\nabla g(x)u).$$

Let us now compare it with the standard calculus-free approach from the literature. To the best of our knowledge, the most powerful result was obtained recently by Gfrerer and Mordukhovich in [14, Corollary 5.4] and it states that

$$DN_\Gamma(x,x^*)(u) = \bigcup_{\lambda\in\Lambda(x,x^*;u)} \nabla^2\langle\lambda,g\rangle(x)u + N_{\mathscr{K}_\Gamma(x,x^*)}(u)$$

holds without any additional assumptions. Here $\Lambda(x,x^*;u)$ denotes the set of directional (or maximal) multipliers. Moreover, this result is not limited to the polyhedral setting and with the addition of the suitable *curvature term*, which vanishes in the polyhedral setting, the formula holds for set D being so-called C^2*-cone reducible* at $g(x)$.

Let us briefly comment on the above-mentioned differences.

- *Multipliers.* It is not difficult to show that $D(N_D\circ g)(x,\lambda)(u)\neq\emptyset$ implies $\lambda\in\Lambda(x,x^*;u)$. Hence, despite the impression, there is in fact no difference regarding the multipliers between the two approaches and only the maximal multipliers are used.
- *Additional metric subregularity.* In order to show the equality

  $$D(N_D\circ g)(x,\lambda)(u) = N_{\mathscr{K}_D(g(x),\lambda)}(\nabla g(x)u),$$

  one indeed needs an addition assumption. It is worth of noting, however, that we do not really need these sets to be the same. In particular, for our purposes it is sufficient to show that

  $$\nabla g(x)^T D(N_D\circ g)(x,\lambda)(u) = N_{\mathscr{K}_\Gamma(x,x^*)}(u).$$

- *Polyhedrality.* Of course, our approach is not really limited to the polyhedral case, since we only need the inner calmness* of $\Lambda(x,x^*)$. Thus, we may state our results in very general setting, *even without the convexity*, using the inner calmness* of $\Lambda(x,x^*)$ as an assumption. The question is, however, how often will this assumption be fulfilled in the nonpolyhedral setting.

Finally, as a specific application we derive the following result regarding the semismoothness* of the normal cone mapping.

Proposition 5.2. *If the mapping $(x',\lambda')\rightrightarrows(g(x'),\lambda')-\mathrm{gph}\,N_D$ is metrically subregular at $\big((x,\lambda),(0,0)\big)$ for every $\lambda\in\Lambda(x,x^*)$, then the normal cone mapping $x'\rightrightarrows N_\Gamma(x')$ is semismooth* at (x,x^*), i.e., for every pair of directions $(u,u^*)\in\mathbb{R}^n\times\mathbb{R}^n$ we have*

$$\langle u,w\rangle = \langle u^*,w^*\rangle \quad \forall\,(w^*,w)\in \mathrm{gph}\,D^*N_\Gamma((x,x^*);(u,u^*)).$$

Proof. Clearly, if $u^*\notin DN_\Gamma(x,x^*)(u)$, then $D^*N_\Gamma((x,x^*);(u,u^*))=\emptyset$ and there is nothing to prove. Hence, let $u^*\in DN_\Gamma(x,x^*)(u)$ and consider $w\in D^*N_\Gamma((x,x^*);(u,u^*))(w^*)$. Proposition 5.1 yields the existence of $\lambda\in\Lambda(x,x^*)$ and $\eta\in D\Lambda((x,x^*),\lambda)(u,u^*)$, in particular,

$$\eta\in D(N_D\circ g)(x,\lambda)(u) \;\text{ and }\; \nabla^2\langle\lambda,g\rangle(x)u+\nabla g(x)^T\eta = u^*,$$

with

$$w\in\nabla^2\langle\lambda,g\rangle(x)w^* + D^*(N_D\circ g)((x,\lambda);(u,\eta))(\nabla g(x)w^*).$$

Hence

$$\langle u,w\rangle = \langle u,\nabla^2\langle\lambda,g\rangle(x)w^*\rangle + \langle\eta,\nabla g(x)w^*\rangle = \langle\nabla^2\langle\lambda,g\rangle(x)u+\nabla g(x)^T\eta,w^*\rangle = \langle u^*,w^*\rangle,$$

provided the mapping $N_D \circ g$ is semismooth* at (x,λ).

The semismoothness* of $N_D \circ g$, however, follows from

$$D^*(N_D \circ g)((x,\lambda);(u,\eta))(\xi^*) \subset \nabla g(x)^T D^* N_D((g(x),\lambda);(\nabla g(x)u,\eta))(\xi^*),$$

which holds due to the assumed metric subregularity, together with the fact that the (polyhedral) map $(y,\lambda) \rightrightarrows N_D(y,\lambda)$ is semismooth* at every point of its graph [18, page 7]. Indeed, given $\xi \in D^*(N_D \circ g)((x,\lambda);(u,\eta))(\xi^*)$, there exist $\zeta \in D^* N_D((g(x),\lambda);(\nabla g(x)u,\eta))(\xi^*)$ with $\xi = \nabla g(x)^T \zeta$ and hence

$$\langle u,\xi\rangle = \langle u,\nabla g(x)^T \zeta\rangle = \langle \nabla g(x)u,\zeta\rangle = \langle \eta,\xi^*\rangle$$

by the semismoothness* of mapping N_D. □

Remark 5.3. *We decomposed the argument into two steps to emphasize their independence. On one hand, one can see that the assumed metric subregularity can be replaced by just asking directly for the semismoothness* of $N_D \circ g$. More importantly, it shows that the proof with trivial adjustments can be used to obtain also the parametrized versions of this result from Propositions 5.4 and 5.6.*

5.2 Parametrized constraints

Given again a convex polyhedral set $D \subset \mathbb{R}^s$ and twice continuously differentiable $g : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^s$, consider now the following parameter-dependent constraints $g(p,x) \in D$ and denote

$$\Gamma(p) := \{x \in \mathbb{R}^n \mid g(p,x) \in D\}. \tag{5.25}$$

Here, given a reference point $(\bar{p},\bar{x})$ with $g(\bar{p},\bar{x}) \in D$, instead of metric subregularity we require the existence of $\kappa > 0$ such that for all (p,x) belonging to a neighborhood of $(\bar{p},\bar{x})$ the inequality

$$\operatorname{dist}(x,\Gamma(p)) \leq \kappa \operatorname{dist}(g(p,x),D)$$

holds. This yields, in particular, that for all $(p,x) \in \operatorname{gph}\Gamma$ sufficiently close to $(\bar{p},\bar{x})$ the mapping $g(p,\cdot) - D$ is metrically subregular at $(x,0)$ with modulus κ and hence

$$N_{\Gamma(p)}(x) = \beta(p,x)^T N_D(g(p,x))$$

for $\beta(p,x) = \nabla_2 g(p,x)$. Moreover, given $x^* \in N_{\Gamma(p)}(x)$, for every

$$\lambda \in \Lambda((p,x),x^*) = \{\lambda \in N_D(g(p,x)) \mid \beta(p,x)^T \lambda = x^*\}$$

we have

$$\mathcal{K}_{\Gamma(p)}(x,x^*) = \beta(p,x)^{-1}\mathcal{K}_D(g(p,x),\lambda), \qquad N_{\mathcal{K}_{\Gamma(p)}(x,x^*)}(u) = \beta(p,x)^T N_{\mathcal{K}_D(g(p,x),\lambda)}(\beta(p,x)u).$$

Finally, since $\operatorname{gph}\Gamma = \{(p,x) \mid g(p,x) \in D\}$ and $\operatorname{dist}((p,x),\operatorname{gph}\Gamma) \leq \operatorname{dist}(x,\Gamma(p))$, we conclude that the mapping $g(\cdot) - D$ is metrically subregular at $\big((p,x),0\big)$ for every $(p,x) \in \operatorname{gph}\Gamma$ sufficiently close to $(\bar{p},\bar{x})$. Therefore, denoting $y = (p,x)$, we have

$$T_{\operatorname{gph}\Gamma}(y) = \nabla g(y)^{-1} T_D(g(y)), \qquad N_{\operatorname{gph}\Gamma}(y) = \nabla g(y)^T N_D(g(y))$$

and hence the critical cone to $\operatorname{gph}\Gamma$ can be also expressed via the critical cone to D in the obvious manner. In particular, given $\lambda \in N_D(g(y))$ with $\nabla g(y)^T \lambda \in N_{\operatorname{gph}\Gamma}(y)$, for every $v = (q,u)$ we obtain

$$N_{\mathcal{K}_{\operatorname{gph}\Gamma}(y,\nabla g(y)^T \lambda)}(v) = \nabla g(y)^T N_{\mathcal{K}_D(g(y),\lambda)}(\nabla g(y)v).$$

Note that we again arrive at the analogous product structure as in the nonparametrized setting. Hence, the product rule from Theorem 4.20 can be applied, taking into account that the map $(y,x^*) \rightrightarrows \Lambda(y,x^*)$, inner calm* by Theorem 3.10, again acts as the "intermediate" mapping. The corresponding auxiliary result for the graphical derivative, together with the same arguments as those after Proposition 5.1, give

$$\begin{aligned} DN_{\Gamma(p)}(y,x^*)(v) &= \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle \lambda, \beta \rangle(y)v + \beta(y)^T D(N_D \circ g)(y,\lambda)(v), \\ &\subset \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle \lambda, \beta \rangle(y)v + \beta(y)^T N_{\mathscr{K}_D(g(y),\lambda)}(\nabla g(y)v). \end{aligned}$$

Since $\nabla g(y)v = \nabla_1 g(y)q + \beta(y)u$ and

$$N_{\mathscr{K}_D(g(y),\lambda)}(\nabla g(y)v) = \big(\mathscr{K}_D(g(y),\lambda)\big)^\circ \cap \big(\nabla_1 g(y)q + \beta(y)u\big)^\perp,$$

in case of the "weak parametrization" ($\operatorname{rg} \nabla_1 g(y) = \{0\}$), we obtain same result as in the nonparametrized setting. In fact, the weak parametrization can be relaxed to condition $\operatorname{rg} \nabla_1 g(y) \subset \operatorname{lin} T_D(g(y))$.

With no additional assumptions, we have two ways how to proceed. First, simply forgetting the argument $\nabla g(y)v$, i.e., replacing it by 0, yields

$$DN_{\Gamma(p)}(y,x^*)(v) \subset \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle \lambda, \beta \rangle(y)v + \big(\mathscr{K}_{\Gamma(p)}(x,x^*)\big)^\circ, \tag{5.26}$$

and we can restrict the multipliers to those fulfilling $\langle \lambda, \nabla g(y)v \rangle = 0$ due to $\nabla g(y)v \in \mathscr{K}_D(g(y),\lambda)$. On the other hand, one can also provide an estimate in terms of the graph of Γ. Indeed, taking into account that $\nabla g(y)^T = (\nabla_1 g(y), \beta(y))^T$ we obtain

$$DN_{\Gamma(p)}(y,x^*)(v) \subset \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle \lambda, \beta \rangle(y)v + \pi_2\Big(N_{\mathscr{K}_{\operatorname{gph}\Gamma}(y,\nabla g(y)^T\lambda)}(v)\Big), \tag{5.27}$$

where $\pi_2 : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the projection operator given by $\pi_2(p,x) = x$.

The aforementioned paper [14] by Gfrerer and Mordukhovich actually deals with the same parametrized setting (5.25), while the results for the nonparametrized setting are just a special case of the weak parametrization. Firstly, based on [14, Lemma 4.1(i)], one can infer that (5.26) holds without the polyhedrality assumption in the C^2-cone reducible case, with the addition of the suitable curvature term, naturally. On the other hand, the estimate (5.27) holds either in the polyhedral setting, or under $\operatorname{rg} \nabla_1 g(y) \subset \operatorname{rg} \beta(y) + \operatorname{lin} T_D(g(y))$, see [14, Theorem 5.3]. Moreover, in the C^2-cone reducible case, the opposite inclusion in (5.27) follows without additional assumptions from [14, Theorem 3.3].

Finally, the analogous arguments as used in the proof of Proposition 5.2 yield the following parametrized version, see also Remark 5.3.

Proposition 5.4. *Suppose that the mapping $(y',\lambda') \rightrightarrows \big(g(y'),\lambda'\big) - \operatorname{gph} N_D$ is metrically subregular at $\big((y,\lambda),(0,0)\big)$ for every $\lambda \in \Lambda(y,x^*)$. Then the normal cone mapping $(p',x') \rightrightarrows N_{\Gamma(p')}(x')$ is semismooth* at $(y,x^*) = (p,x,x^*)$.*

5.3 Constraints depending on the parameter and the solution

Let $D \subset \mathbb{R}^s$ be again convex polyhedral, consider twice continuously differentiable mapping $g : \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^s$ and denote

$$\Gamma(p,x) := \{z \in \mathbb{R}^n \mid g(p,x,z) \in D\}, \tag{5.28}$$

i.e., the feasible set depends also on the solution x. As before, given a reference point $(\bar{p},\bar{x})$ with $g(\bar{p},\bar{x},\bar{x}) \in D$, assume that there exists some $\kappa > 0$ such that for all (p,x,z) belonging to a neighborhood of $(\bar{p},\bar{x},\bar{x})$ the inequality

$$\operatorname{dist}(z,\Gamma(p,x)) \leq \kappa \operatorname{dist}(g(p,x,z),D) \tag{5.29}$$

holds. Denoting $\beta(p,x) = \nabla_3 g(p,x,x)$ and $\tilde{g}(p,x) = g(p,x,x)$, we obtain

$$N_{\Gamma(p,x)}(x) = \beta(p,x)^T N_D(\tilde{g}(p,x)) = \beta(y)^T N_D(\tilde{g}(y))$$

for $y = (p,x)$. On top of it, all the analogous relations for computing the normals, tangents and critical directions remain valid.

Given $v = (q,u) \in \mathbb{R}^l \times \mathbb{R}^n$ and $w^* \in \mathbb{R}^n$, Theorem 4.20 yields

$$DN_{\Gamma(p,x)}(y,x^*)(v) = \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle\lambda,\beta\rangle(y)v + \beta(y)^T D(N_D \circ \tilde{g})(y,\lambda)(v)$$

and for $u^* \in DN_{\Gamma(p,x)}(y,x^*)(v)$ also

$$\begin{aligned} &D^*N_{\Gamma(p,x)}((y,x^*);(v,u^*))(w^*) \\ &\quad \subset \bigcup_{\lambda \in \Lambda(y,x^*)} \bigcup_{\eta \in D\Lambda((y,x^*),\lambda)(v,u^*)} \nabla\langle\lambda,\beta\rangle(y)^T w^* + D^*(N_D \circ \tilde{g})((y,\lambda);(v,\eta))(\beta(y)w^*). \end{aligned}$$

Proceeding as in the previous section, we can immediately derive two estimates for the graphical derivative as follows. First, applying the calculus to $N_D \circ \tilde{g}$ as well as the reduction lemma as before, and then replacing the argument $\nabla\tilde{g}(y)v$ by 0, yields

$$DN_{\Gamma(p,x)}(y,x^*)(v) \subset \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle\lambda,\beta\rangle(y)v + \left(\mathscr{K}_{\Gamma(p,x)}(x,x^*)\right)^\circ,$$

where $\langle\lambda,\nabla\tilde{g}(y)v\rangle = 0$. On the other hand, from $\nabla g(y)^T = (\nabla_1 g(y),\nabla_2 g(y),\beta(y))^T$ we infer

$$DN_{\Gamma(p)}(y,x^*)(v) \subset \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle\lambda,\beta\rangle(y)v + \pi_3\left(N_{\mathscr{K}_{\operatorname{gph}\Gamma}(y,\nabla g(y)^T\lambda)}(v)\right),$$

where $\pi_3 : \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the projection operator given by $\pi_3(p,x,z) = z$.

Moreover, we obtain the following more delicate result.

Theorem 5.5. *For every $v \in \mathbb{R}^l \times \mathbb{R}^n$ we have*

$$DN_{\Gamma(p,x)}(y,x^*)(v) \subset \bigcup_{\lambda \in \Lambda(y,x^*)} \nabla\langle\lambda,\beta\rangle(y)v + \beta(y)^T N_{\mathscr{K}_D(g(y),\lambda)}(\nabla g(y)v).$$

On the other hand, given $v \in \mathbb{R}^l \times \mathbb{R}^n$, $\lambda \in \Lambda(y,x^*)$ and $\eta \in N_{\mathscr{K}_D(\tilde{g}(y),\lambda)}(\nabla\tilde{g}(y)(v))$, assume that the mapping $F : \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^s \rightrightarrows \mathbb{R}^s \times \mathbb{R}^s$ given by

$$F(y',\lambda') := \left(\tilde{g}(y'),\lambda'\right) - \operatorname{gph} N_D \tag{5.30}$$

is metrically subregular in direction (v,η) at $\left((y,\lambda),(0,0)\right)$. Then we have

$$\nabla\langle\lambda,\beta\rangle(y)v + \beta(y)^T\eta \in DN_{\Gamma(p,x)}(y,x^*)(v)$$

and, moreover, for $u^* \in DN_{\Gamma(p,x)}(y,x^*)(v)$ and arbitrary $w^* \in \mathbb{R}^n$ one has the estimate

$$\begin{aligned} &D^*N_{\Gamma(p,x)}((y,x^*);(v,u^*))(w^*) \\ &\quad \subset \bigcup_{\lambda \in \Lambda(y,x^*)} \bigcup_{\eta \in D\Lambda((y,x^*),\lambda)(v,u^*)} \nabla\langle\lambda,\beta\rangle(y)^T w^* + \nabla\tilde{g}(y)^T D^*N_D((\tilde{g}(y),\lambda);(\nabla\tilde{g}(y)v,\eta))(\beta(y)w^*). \end{aligned}$$

The model was investigated already in [30] by using the classical calculus of Mordukhovich.

Recently, in [17], Gfrerer and Outrata computed the graphical derivative and estimated the directional limiting coderivative of the normal-cone mapping $N_{\Gamma(p,x)}$. The latter task was done using similar calculus rules. In order to proceed, however, they imposed much stronger assumption that guarantees the uniqueness of the multiplier λ and the "direction" η. The underlying reason was that, without explicitly mentioning it, they were in fact relying on the stronger property of inner calmness, see, e.g., [17, Lemmas 4.1 and 4.2].

The huge difference between these restrictive, *nondegeneracy-type* assumptions on one hand, and our mild, *subregularity-like* inequality (5.29) on the other, signifies the gap between the inner calmness and the inner calmness*.

On the other hand, in [3], the authors worked with the same assumption (5.29) to obtain also the same result for the graphical derivative. The explicit estimation of the coderivative was, however, bypassed by addressing the related stability issues directly, see [3, Theorem 6.1].

To sum up, our new calculus approach unifies the computation of the graphical derivative and the estimation of the directional limiting coderivative and it also provides the estimate for the directional limiting coderivative under (5.29). Note that in this case of the constraints depending also on the solution, which is arguably the most challenging case, we actually recover the best known results. We believe that our approach brings an important insight into the demanding proofs and computations. For instance, in [3, 17], the formula for the graphical derivative is derived from [14, Theorem 5.3]. Hence, in order to understand it, one has to go through much more laborious material than actually needed. Our paper clarifies that and offers considerably simpler arguments.

We conclude this section by the corresponding semismoothness* result.

Proposition 5.6. *If the mapping* (5.30) *is metrically subregular at* $\big((y,\lambda),(0,0)\big)$ *for every* $\lambda \in \Lambda(y,x^*)$*, then the normal cone mapping* $(p',x') \rightrightarrows N_{\Gamma(p',x')}(x')$ *is semismooth* at* $(y,x^*) = (p,x,x^*)$.

Conclusion

Looking closely at the remarkable results regarding computation of the graphical derivative of the normal cone mapping, we provide new insights by revealing the underlying calculus. Although the main purpose is to better understand these known results and to simplify and unify their proofs, our attempt yields also several novelties. Indeed, we propose the new calculus rules, in particular the chain rule for the graphical derivative, as well as the new property of inner calmness*. We believe that this paper has already shown that the inner calmness* is very interesting and we also believe that it deserves further study.

We point out that our calculus-based approach does not fully cover the result obtained by the direct approach. Hence, it seems that, when dealing with generalized derivatives, it is not good to rely solely on the calculus rules. On the other hand, we also believe that the calculus provides very solid foundation and offers a simpler, unified view. Therefore, we think it is still useful to look whether new results contain some calculus principles, either the well-established ones or some new, yet-to-be-formulated ones.

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