T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 01 Tuesday, 12 March 2019, Time: 10:15 – 11:45, Room: S3 047.

1 Variational formulation of multi-dimensional elliptic Boundary Value Problems (BVP)

1.1 Scalar Second-order Elliptic BVP

In Section 1.2.1 of our lectures, we considered the BVP in classical formulation

\[
\begin{align*}
\text{Find } u &\in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : \\
&- \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u(x) = f(x), x \in \Omega \\
\text{BC: } &\quad u(x) = g_1(x), x \in \Gamma_1, \\
&\quad \frac{\partial u}{\partial N} := \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u(x)}{\partial x_j} n_i(x) = g_2(x), x \in \Gamma_2, \\
&\quad \frac{\partial u}{\partial N} + \alpha(x) u(x) = g_3(x), x \in \Gamma_3.
\end{align*}
\]

and derived the variational formulation

\[
\begin{align*}
\text{Find } u &\in V_g \text{ such that } a(u, v) = < F, v > \quad \forall v \in V_0, \\
\text{with} \\
a(u, v) &:= \int_{\Omega} \left( \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} v + cuv \right) dx + \int_{\Gamma_3} \alpha uv ds, \\
<F, v> &:= \int_{\Omega} fv dx + \int_{\Gamma_2} g_2 v ds + \int_{\Gamma_3} g_3 v ds, \\
V_g &:= \{v \in V = W^1_2(\Omega) : v = g_1 \text{ on } \Gamma_1 \}, \\
V_0 &:= \{v \in V : v = 0 \text{ on } \Gamma_1 \}.
\end{align*}
\]
under the assumptions

\[ 1) \; a_{ij}, b_i, c \in L_\infty(\Omega), \alpha \in L_\infty(\Gamma_3), \]
\[ 2) \; f \in L_2(\Omega), g_i \in L_2(\Gamma_i), i = 2, 3, \]
\[ 3) \; g_1 \in H^\frac{1}{2}(\Gamma_1), \text{i.e., } \exists \tilde{g}_1 \in H^1(\Omega): \tilde{g}_1|_{\Gamma_1} = g_1. \]
\[ 4) \; \Omega \subset \mathbb{R}^d(\text{bounded}) : \Gamma = \partial \Omega \subset C^{0,1} (\text{Lip boundary}), \]
\[ 5) \; \text{uniform ellipticity:} \]
\[ \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq \bar{\mu}_1 |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \]
\[ a_{ij}(x) = a_{ji}(x) \quad \forall i, j = 1, d \quad \forall \text{a.e. } x \in \Omega. \]

\[ \tag{3} \]

01. Formulate the classical assumptions on \{ \{ a_{ij}, b_i, c, \alpha, f, g_i, \Omega \text{ resp. } \partial \Omega \} \} for (1)!

02. Show that, for sufficiently smooth data, the generalized solution \( u \in V_g \cap X \cap H^2(\Omega) \) of the Boundary Value Problem (2) is also a classical solution, i.e. a solution of (1)!

\[ \begin{cases} 
\text{Find } u \in X = C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : \\
-\Delta u(x) + c(x) u(x) = f(x), x \in \Omega \subset \mathbb{R}^d (\text{bounded}), \\
u(x) = g_1(x), \; x \in \Gamma_1, \\
\frac{\partial u}{\partial n}(x) = g_2(x), \; x \in \Gamma_2, \\
\frac{\partial u}{\partial n}(x) = \alpha(x)(g_3(x) - u(x)), \; x \in \Gamma_3
\end{cases} \]

where \( V_0 = \{ v \in V = H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \} \).

03. Show that the assumptions of the Lax-Milgram-Theorem are satisfied for the variational problem (2) under the assumptions (3) and the additional assumptions \( b_i = 0, \) \( c(x) \geq 0 \) for almost all \( x \in \Omega, \) \( \alpha(x) \geq \alpha = \text{const} > 0 \) for almost all \( x \in \Gamma_3, \) and \( \text{meas}_{d-1}(\Gamma_i) > 0, i = 1, 2, 3! \) What happens in the case \( \Gamma_1 = \emptyset, \) and what happens in the case \( \alpha = 0? \)

04. In addition to assumption (3), let us assume that \( c(x) \geq \zeta = \text{const} > 0 \) for almost all \( x \in \Omega, \) \( \Gamma_1 = \Gamma_3 = \emptyset, \) and \( b_i \neq 0. \) Provide conditions for the coefficients \( b_i(\cdot) \) such that the assumptions of the Lax-Milgram-Theorem are satisfied!

\( \circ \) Hint: For the estimate of the convection term \( \sum_{i=1}^{d} \int_{\Omega} b_i \frac{\partial u}{\partial x_i} v \, dx, \) make use of the \( \varepsilon \)-inequality (Young’s inequality)
\[ |ab| \leq \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2, \quad \forall a, b \in \mathbb{R}^1, \quad \forall \varepsilon > 0! \]
Derive the variational formulation of the pure Neumann problem for the Poisson equation

\[-\Delta u = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma := \partial \Omega, \quad (4)\]

and discuss the question of the existence and uniqueness of a generalized solution of (4)!

**Hint:**
Obviously, \( u(x) + c \) with an arbitrary constant \( c \in \mathbb{R}^1 \) solves (4) provided that \( u \) is the solution of the BVP (4). There are the following ways to analyze the existence of a generalized solution:

1) Set up the variational formulation in \( V = H^1(\Omega) \) and apply the **Fredholm-Theory**!
2) Set up the variational formulation in the factor-space \( V = H^1(\Omega)|_{\ker} \) with \( \ker = \{ c : c \in \mathbb{R}^1 \} = \mathbb{R}^1 \) and apply the **Lax-Milgram-Theorem**!

**Hint:**
Apply the Fredholm theory to the operator equation

Find \( u \in V_0 : (I - K)u = \tilde{f} \text{ in } V_0 \)

that arises from the setting

\[
\int_\Omega \left[ \nabla^T u \nabla v + uv \right] dx - (1 + \omega^2) \int_\Omega uv dx = \int_\Omega f v dx
\]

which is equivalent to the variational formulation of the Helmholtz equation.

**Hint:**}
Apply the Fredholm theory to the operator equation

Find \( u \in V_0 : (I - K)u = \tilde{f} \text{ in } V_0 \)