Road Map for Preparing the Oral Examination

1. Derive the variational formulation of the second-order boundary value problem (5)

\[-\text{div}(A\nabla u) + b \cdot \nabla u + cu = f \text{ in } \Omega,\]
\[u = g_1 \text{ on } \Gamma_1,\]
\[\frac{\partial u}{\partial N} := (A\nabla u) \cdot n = g_2 \text{ on } \Gamma_2,\]
\[\frac{\partial u}{\partial N} + \alpha u = g_3 \text{ on } \Gamma_3,\]

given in Subsection 1.2.1, and show existence and uniqueness under some assumptions specified by the examiner!

2. Derive the variational formulation of the linear elasticity problem (8) given in Subsection 1.2.2, and show existence and uniqueness under some assumptions specified by the examiner!

3. Derive the variational formulation of the first biharmonic BVP (12) given in Subsection 1.2.3, and show existence and uniqueness of a weak solution in \(H^2_0(\Omega)\)? Which combinations of boundary conditions are possible?

4. Formulate and prove Sobolev’s norm equivalence theorem (Theorem 2.13)! Use it for proving Friedrichs- and Poincaré-type inequalities!

5. Formulate and prove the Lemma of Bramble and Hilbert (Lemma 2.17)!

6. Derive the formula of integration by parts and other famous integrations formula from the main formula of the differential and integral calculus in an appropriate Sobolev space setting! How can you use the formula of integration by parts to derive trace theorem using the example of \(H(div)(\Omega)\)? (Theorem 2.19)!

7. What do you know about mesh generation, regular meshes, and internal representation of the mesh with appropriate files? Provide a mesh and its internal representation of some sample domains given by the examiner! Give the definition of the FE Nodal Basis and of the \(V_h, V_{0h}, V_{gh}\) for a given triangular mesh via mapping technique!
8. Describe the generation of the FE equations \( K_h u_h = f_h \) via the three steps
   a) assembling of the load vector,
   b) assembling of the stiffness matrix,
   c) incorporating the boundary conditions!

9. Describe the properties of the system of finite element equations and estimate the spectral condition number of the stiffness matrix \( K_h \) in the SPD case!

10. Prove the approximation error estimate

\[
\inf_{v_h \in V_h} |u - v_h|_{H^s(\Omega)} \leq ?
\]

for \( s = 0 \) or \( s = 1 \), where \( H^0(\Omega) = L^2(\Omega) \)!

11. Prove the \( H^1 \)-Convergence of the FE-solutions \( u_h \) to the solution \( u \) of a \( V_0 \)-elliptic and \( V_0 \)-bounded elliptic BVP!

12. How can you prove an optimal \( L^2 \) convergence rate estimate for the FE-solutions \( u_h \) to the solution \( u \) of a \( V_0 \)-elliptic and \( V_0 \)-bounded elliptic BVP!

13. What do you know about the \( L^\infty \)-convergence of FE-solutions!

14. Prove the first Lemma of Strang! What are the typical applications of the first Lemma of Strang?

15. Prove the second Lemma of Strang! What are the typical applications of the second Lemma of Strang?

16. Explain the Clément interpolation operator and show its approximation properties!

17. Derive the residual error estimator for the homogeneous Dirichlet BVP for the Poisson equation! How do you have to modify the residual error estimator in the case of mixed boundary conditions?

18. Derive the residual error estimator for our heat conduction problem “CHIP”!

19. Derive and discuss Repin’s functional a posteriori error estimator for the homogeneous Dirichlet BVP for the Poisson equation!

20. Derive dG variational formulations and dG schemes for the homogenous Dirichlet problem for the Poisson equation

\[-\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2 \text{ and } u = 0 \text{ on } \Gamma = \partial \Omega. \quad (1)\]
Show that, under suitable conditions, the SIPG bilinear form \( a_h(\cdot, \cdot) \) is \( V_k(T_h) \) elliptic and bounded! How can you estimate the discretization error in the DG norm?

21. Derive dG variational formulations and dG schemes for the CHIP problem! Show that, under suitable conditions, the SIPG bilinear form \( a_h(\cdot, \cdot) \) is \( V_k(T_h) \) elliptic and bounded! How can you estimate the discretization error in the DG norm?

22. Demonstrate the ideas of the classical Finite Difference Method (FDM) on rectangular grids and the more advanced Finite Volume Method (FVM) on triangular grids for the Poisson equation \(-\Delta u = f\) in \( \Omega \in \mathbb{R}^2 \) under Dirichlet boundary conditions \( u = g \) on \( \Gamma = \partial\Omega \) ! Show that discrete convergence follows from stability and approximation!

All Exercises given explicitly in lectures are also a subject of the Oral Examination !!!