Summary: Multilevel ASM preconditioners

that are uniquely defined by

1. the multilevel space splitting $\mathbb{V} = \sum_{j=1}^{J} \mathbb{V}_j$ i.a. splitting index ≠ level index
2. $a_j(\cdot,\cdot): \mathbb{V}_j \times \mathbb{V}_j \to \mathbb{R}$,

namely

$$\tilde{P} u = \sum_{j=1}^{J} \tilde{P}_j u = \sum_{j=1}^{J} \Phi V_j \underline{u}_j = \sum_{j=1}^{J} \Phi V_j C_j^{-1} V_j^T K \underline{u} = \Phi \sum_{j=1}^{J} V_j C_j^{-1} V_j^T K \underline{u}$$

Φ

$$\underline{u}_j \in \mathbb{V}_j : a_j(\Phi V_j \underline{u}_j, \Phi V_j \underline{v}_j) = a(u, \Phi V_j \underline{v}_j) \quad \forall v_j = \Phi V_j \underline{v}_j \in \mathbb{V}_j$$

$$C_j \underline{u}_j = V_j^T K \underline{u} \qquad \underline{v}_j \in \mathbb{R}^{n_j}$$

$$\underline{\tilde{P} u} = C^{-1} K \underline{u} = \sum_{j=1}^{J} V_j C_j^{-1} V_j^T K \underline{u}$$

inexact

exact

No	name	splitting $\mathbb{V} = \mathbb{V}_L = \sum_{j=1}^{J} \mathbb{V}_j$	$a_j(\cdot,\cdot)$	$\kappa(\tilde{P}) = \kappa(C^{-1}K)$	$ops(C^{-1}\underline{d})$
1	is practically meaningless	$\mathbb{V} = \sum_{l=0}^{L} \mathbb{V}_l$	$a_l(\cdot,\cdot) := a(\cdot,\cdot)$	$O(L)$	?
2	Multilevel L_2-projections	$\mathbb{V} = \sum_{l=0}^{L} \mathbb{V}_l$	$a_l(\cdot,\cdot) := 2^{2(l-1)}(\cdot,\cdot)_{L_2}$	$O(1)$	$M_j \to D_j$ $O(N_L)$
3	BPX (1990)	$\mathbb{V} = \sum_{l=0}^{L} \sum_{j=1}^{J_l} \mathbb{V}_{l,j}$	$a_{l,j}(\cdot,\cdot) := 2^{2(l-1)}(\cdot,\cdot)_{L_2}$	$O(1)$	$O(N_L)$
4	MDS (1992)	$\mathbb{V} = \sum_{l=0}^{L} \sum_{j=1}^{J_l} \mathbb{V}_{l,j}$, $j \in \omega_l$	$a_{l,j}(\cdot,\cdot) := a(\cdot,\cdot)$	$O(1)$	$O(N_L)$
5	HB (1986)	$\mathbb{V} = \sum_{l=0}^{L} \sum_{j \in \omega_l \setminus \omega_{l-1}} \mathbb{V}_{l,j}$	$a_{l,j}(\cdot,\cdot) := 2^{2(l-1)}(\cdot,\cdot)_{L_2}$	$O(1)$, $d=1$; $O(L^2)$, $d=2$; $O(h_L^{-1})$, $d=3$	$O(N_L)$
⋮	e.g. wavelets	⋮	⋮	⋮	⋮

where $\mathbb{V}_{l,j} = \mathrm{span}\{\varphi_{l,j}\} = \mathrm{span}\{\Phi V_{l,j}\}$, $\dim \mathbb{V}_{l,j} = 1$, $\omega_{-1} = \emptyset$,

BPX = Bramble-Pasciak-Xu (1990)

MDS = Multilevel Diagonal Scaling (1992) by Zhang,

HB = Hierarchical Basis (1986) by H. Yserentant.