Theorem 3.23: (Inexact MSM convergence)

A.: Let the Assumptions 3.18 - 3.20 be fulfilled, and let ω from Assumption 3.20 be contained in $(0,2)$.

S.: Then the error propagation operator E of the inexact MSM satisfies the estimate:

$$(45) \quad \|E\|_a^2 = \|E_{MSM}\|_a^2 \le 1 - \frac{2-\omega}{\max\{1,\omega\}^2 \varrho(\Gamma)^2 c_L^2} = q_{MSM}^2 < 1,$$

where the constants c_L and $\varrho(\Gamma)$ are defined in Assumptions 3.18 and 3.18, respectively.

Proof:

- Let us define

$$\tilde{E}_j = (I-\tilde{P}_j)\cdot\ldots\cdot(I-\tilde{P}_1), \; j=\overline{1,J}, \quad E_0 := I,$$

$$Q_j = 2P_j - P_j^2 = (2I-P_j)P_j, \quad j=\overline{1,J}.$$

Then we have the recurrancy

$$(46) \quad E_{j-1}^* E_{j-1} - E_j^* E_j = E_{j-1}^* Q_j E_{j-1}, \quad j=\overline{1,J},$$

where star "$*$" means the adjoint wrt $a(\cdot,\cdot)$, i.e

$$a(E_j u, v) = a(u, E_j^* v) \quad \forall u,v \in V.$$

Indeed,

$$E_{j-1}^* E_{j-1} - E_j^* E_j = E_{j-1}^* E_{j-1} - E_{j-1}^*(I-\overset{sa}{\cancel{P_j^*}})(I-P_j)E_{j-1}$$

$$= E_{j-1}^* \underbrace{(I-(I-2P_j+P_j^2))}_{=(2I-P_j)P_j = Q_j} E_{j-1}$$

$$= E_{j-1}^* Q_j E_{j-1}.$$