Special Lecture on

Boundary Element Methods

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Contents

Chapter 1

Introduction

The boundary element method (BEM) is a numerical method for solving partial differential equations (PDEs), based on the following steps.

1. Reformulation of the PDE as *boundary integral equation(s)*. These are integral equations where the unknown only lives on the boundary of the computational domain, and they are constructed using the *fundamental solution* of the underlying differential operator.

2. *Discretize* the boundary integral equations.

3. *Solve* the discrete equations.

4. If necessary, *reconstruct* a quantity of interest from the discrete solution, e.g. the solution inside the domain from the (approximated) Cauchy data on the boundary.

In this introduction, the BEM approach is *sketched* in a couple of examples (without going into details). This should provide an overview on the whole lecture.

Model Problem To show the main ideas of BEM, we consider a simple model problem. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded domain with boundary $\Gamma := \partial\Omega$. We want to find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega\,, \\ u &= g_D \quad \text{on } \Gamma\,, \end{aligned} \tag{1.1}$$

for a given Dirichlet trace g_D, i.e., we want to solve the Dirichlet boundary value problem for Laplace's equation.

Fundamental Solution The *fundamental solution* to the Laplace operator $-\Delta$ is given by

$$U^*(x,\, y) = \begin{cases} -\frac{1}{2\pi} \log|x-y| & \text{for } d = 2\,, \\ \frac{1}{4\pi} \frac{1}{|x-y|} & \text{for } d = 3\,. \end{cases}$$

A characteristic property of this fundamental solution is that

$$-\Delta_x U^*(x,\, y) \;=\; \delta_y(x),$$

where δ_y denotes the Dirac delta distribution. Furthermore,

$$-\Delta_x U^*(x,\, y) \;=\; 0 \quad \forall y \neq x$$

in the strong sense (note that $U^*(x,\, y)$ is C^∞ unless $x = y$).

■ Boundary Integral Equations

Indirect Approach We make the following *ansatz* for the solution $u : \overline{\Omega} \to \mathbb{R}$ of (1.1):

$$u(x) \;=\; \underbrace{\int_\Gamma U^*(x,\, y)\, w(y)\, ds_y}_{=:(\widetilde{V}w)(x),\ \textit{single layer potential}} \qquad \text{for } x \in \Omega\,, \tag{1.2}$$

for some (yet unknown) density $w : \Gamma \to \mathbb{R}$. Indeed, u fulfills the homogeneous Laplace equation (at least for smooth w):

$$-\Delta_x \int_\Gamma U^*(x,\, y) w(y)\, ds_y \;=\; \int_\Gamma \underbrace{-\Delta_x U^*(x,\, y)}_{=0}\, w(y)\, ds_y \;=\; 0 \qquad \forall x \in \Omega$$

(we are allowed to switch integration and differentiation because $x \neq y$). Now, we take the trace of (1.2) for $x \in \Gamma$. As we will show later on (non-trivial!),

$$u(x) \;=\; \underbrace{(Vw)(x) \;:=\; \int_\Gamma U^*(x,\, y)\, w(y)\, ds_y}_{\textit{single layer potential operator}} \quad \text{for } x \in \Gamma\,.$$

Hence, in order to fulfill the Dirichlet boundary conditions in (1.1), the density $w : \Gamma \to \mathbb{R}$ must satisfy

$$Vw \;=\; g_D \qquad \text{on } \Gamma\,. \tag{1.3}$$

This is a boundary integral equation (BIE) of the first kind (see the definition below). From the solution w of this equation, we can reconstruct the solution u of (1.1) using formula (1.2).

Definition 1.1. Let $\mathcal{M}$ be a domain or manifold and let s denote the volume/surface measure. An equation of the form

$$a(x)\,u(x) + \int_{\mathcal{M}} k(x,\,y)\,u(y)\,ds_y \;=\; f(x) \qquad \text{for } x \in \mathcal{M}$$

is called *integral equation* of

- *first kind*, if $a = 0$,
- *second kind*, if $\forall x \in \mathcal{M} : a(x) > 0$ (or $\forall x \in \mathcal{M} : a(x) < 0$).

The operator K defined by

$$(Ku)(x) := \int_{\mathcal{M}} k(x,\,y)\,u(y)\,ds_y \qquad \text{for } x \in \mathcal{M}$$

is called *integral operator* with *(integral) kernel* $k(x,\,y)$.

Direct Approach Green's second identity states that

$$\int_\Omega -u\,\Delta v + v\,\Delta u\,dx \;=\; \int_\Gamma -u\,\frac{\partial v}{\partial n} + v\,\frac{\partial u}{\partial n}\,ds_x$$

for all (sufficiently smooth) $u,\,v : \overline{\Omega} \to \mathbb{R}$, where n is the outward unit normal vector on Γ. Setting (formally) $v(x) := U^*(x,\,y)$ with $y \in \Omega$, we obtain that for all $y \in \Omega$:

$$\begin{aligned}\int_\Omega \underbrace{\big[-\Delta_x U^*(x,\,y)\big]}_{=\delta_y(x)}\,u(x)\,dx + \int_\Omega U^*(x,\,y)\,\Delta u(x)\,dx \\ = \;-\int_\Gamma u(x)\,\frac{\partial}{\partial n_x}U^*(x,\,y)\,ds_x + \int_\Gamma U^*(x,\,y)\,\frac{\partial u}{\partial n}(x)\,ds_x\end{aligned}$$

Without worrying about correctness, we interpret the first integral as an evaluation of the delta distribution δ_y. If u is a solution of the homogeneous Laplace equation in Ω, then

$$\begin{aligned}u(y) \;=\; \int_\Omega U^*(x,\,y)\,\underbrace{\big[-\Delta u(x)\big]}_{=0}\,dx - \int_\Gamma \Big[\frac{\partial}{\partial n_x}U^*(x,\,y)\Big]\,\underbrace{u(x)}_{=\gamma_0 u}\,ds_x \\ + \int_\Gamma U^*(x,\,y)\,\underbrace{\frac{\partial u}{\partial n}(x)}_{=\gamma_1 u}\,ds_x \qquad \forall y \in \Omega\,.\end{aligned} \tag{1.4}$$

Equation (1.4) is called *Green's third identity* or *representation formula*. It represents the value of u in the interior of Ω in terms of the *Cauchy data*

$$\begin{bmatrix} \gamma_0 u \\ \gamma_1 u \end{bmatrix} = \begin{bmatrix} u \\ \frac{\partial u}{\partial n} \end{bmatrix} \qquad \text{on } \Gamma\,,$$

i. e., the *trace* of u and its *normal derivative*.

We now take the trace of (1.4) for $y \in \Gamma$ (again non-trivial, because the singular case $x = y$ is included). The outcome is:

$$u(y) = \big(1 - \sigma(y)\big)u(y) - (K\,\gamma_0 u)(y) + (V\,\gamma_1 u)(y) \qquad \forall y \in \Gamma,$$

where σ is a positive function with values in $(0,\,1)$, K is an integral operator, and V is the single layer potential operator from above. In case of a smooth boundary, $\sigma = 1/2$. For smooth functions v,

$$\underbrace{(K\,v)(x) = \int_\Gamma \Big[\frac{\partial}{\partial n_y} U^*(x,\,y)\Big] v(y)\,ds_y}_{\textit{double layer potential operator}}.^1$$

When inserting the Dirichlet boundary condition $\gamma_0 u = g_D$, we obtain a *first kind BIE* for the unknown normal derivative $t := \gamma_1 u$:

$$Vt = (\sigma I + K) g_D \qquad \text{on } \Gamma. \tag{1.5}$$

The left hand side of (1.5) is of the same form as in (1.3). However, there is a difference: whereas the unknown w in (1.3) is a possibly unphysical function/density, the unknown t in (1.5) is the normal derivative. In many applications, $t = \partial u/\partial n$ is already the quantity of interest.

If needed, the entire solution in Ω can be reconstructed using (1.4).

Further Approaches

1. The indirect approach above was based on the single layer potential. We can also use the *double layer potential ansatz*

$$u(x) = \underbrace{\int_\Gamma \frac{\partial}{\partial n_x} U^*(x,\,y)\,v(y)\,ds_y}_{=:(\widetilde{W}v)(x),\ \textit{double layer potential}} \quad \text{for } x \in \Omega\,,$$

[1] This integral has to be understood in the sense of the Cauchy principal value!

for some density $v : \Gamma \to \mathbb{R}$. One can show that $-\Delta\widetilde{W}v = 0$ in Ω. Taking the trace for $x \in \Gamma$ yields

$$\gamma_0 u \;=\; (-1+\sigma)v + Kv.$$

Inserting the Dirichlet condition, we obtain the *second kind BIE*

$$\big((-1+\sigma)I + K\big)v \;=\; g_D \qquad \text{on } \Gamma.$$

2. Formula (1.5) was derived by taking the trace "γ_0(1.4)". We can also use "γ_1(1.4)". Without going into details, this yields

$$\gamma_1 u \;=\; D\,\gamma_0 u + (\sigma I + K')\gamma_1 u, \tag{1.6}$$

where D and K' are boundary integral operators (called *hypersingular integral operator* and *adjoint double layer potential operator*, respectively). Inserting the Dirichlet condition, we obtain the *second kind BIE*

$$\big((-1+\sigma)I + K'\big)t \;=\; -D\,g_D\,.$$

for the unknown normal derivative $t = \partial u/\partial n$.

3. Similar techniques can be applied for the *Neumann boundary value problem* of Laplace's equation. Two examples:

   (a) We can use the trace of the representation formula (1.4) (explained in the direct approach), insert the given Neumann data $\partial u/\partial n = g_N$, and obtain the *second kind BIE*

   $$(\sigma I + K)\gamma_0 u \;=\; V g_N \qquad \text{on } \Gamma,$$

   for the unknown trace $\gamma_0 u$.

   (b) Inserting the Neumann condition into (1.6), we obtain the *first kind BIE*

   $$D\,\gamma_0 u \;=\; \big((1-\sigma)I - K'\big)g_N \qquad \text{on } \Gamma.$$

An Exterior Problem Often, the PDE is given in an *exterior domain*. Let $\Omega^{\text{int}} \subset \mathbb{R}^d$ be a bounded domain and let $\Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}^{\text{int}}$ denote its complement. The exterior Dirichlet problem for the Laplace equation reads

$$\begin{aligned} -\Delta u \;&=\; 0 && \text{in } \Omega^{\text{ext}}, \\ u \;&=\; g_D && \text{on } \Gamma := \partial\Omega^{\text{ext}}, \\ |u(x)| \;&=\; \mathcal{O}(|x|^{-1}) && \text{as } |x| \to \infty \qquad \textit{(radiation condition).} \end{aligned}$$

Under appropriate assumptions, one can show the following Green identity in Ω^{ext}: if u fulfills the homogeneous Laplace equation in Ω^{ext} and the radiation condition, then

$$u(y) \;=\; \int_\Gamma \Big[\frac{\partial}{\partial n_x} U^*(x,\, y)\Big] u(x)\, ds_x - \int_\Gamma U^*(x,\, y)\, \frac{\partial u}{\partial n}(x)\, ds_x \qquad \forall y \in \Omega^{\text{ext}},$$

where n is the unit normal vector on Γ pointing into Ω^{ext} (i.e., outward to Ω^{int}). Taking the trace yields

$$u(y) \;=\; (1-\sigma(y))u(y) + (K\,\gamma_0^{\text{ext}} u)(y) - (V\gamma_1^{\text{ext}} u)(y) \qquad \forall y \in \Gamma,$$

where $\gamma_0^{\text{ext}} u$ denotes the exterior trace of u at Γ and $\gamma_1^{\text{ext}} u = \partial u/\partial n$ the exterior normal derivative. Inserting the Dirichlet condition yields the *first kind BIE*

$$Vt \;=\; (-\sigma I + K) g_D \qquad \text{on } \Gamma,$$

for the (exterior) normal derivative $t = \partial u/\partial n$.

Rigorous Study In this lecture, we clarify in which (Sobolev) spaces we have to work and how to derive the BIEs. Our starting point will be the weak formulation of an elliptic PDE, and we will need to use the theory of distributions. Furthermore, we will study the (unique) solvability of the BIEs. This requires a closer inspection of the four boundary integral operators V, K, K', and D that appeared above.

■ Discretization

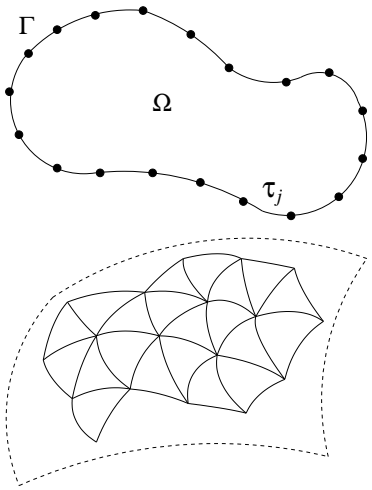
All the previous BIEs are of the form

$$B\,v \;=\; g \qquad \text{on } \Gamma, \tag{1.7}$$

where B is an operator mapping functions on Γ to functions on Γ. In order to discretize, one considers a *mesh* of the boundary Γ:

$$\Gamma \;=\; \bigcup_{j=1}^{n} \tau_j\,.$$

Two approaches are common: *collocation* and *Galerkin BEM*.



Collocation BEM In the *collocation method*, one considers approximations v_h of the form

$$v_h(x) \;=\; \sum_{j=1}^{n} v_j \, \chi_{\tau_j} \,,$$

where χ_{τ_j} denotes the characteristic function. Of course, we cannot expect $B\, v_h = g$ to hold on the whole boundary. Instead, we choose a set of test points $\{y_j\}_{j=1\ldots n} \subset \Gamma$ (e.g., with $y_j \in \tau_j$) such that

$$(B\, v_h)(y_i) \;=\; g(y_i) \qquad \forall i = 1 \ldots n \,.$$

This leads to a linear system of equations for the coefficients $[v_j]_{j=1}^n$.

Galerkin BEM In the *Galerkin BEM*, we first rewrite (1.7) as a variational problem. Let X be a Hilbert space of function where the solution v is sought. In many cases, $B : X \to X^*$ and $g \in X^*$. Then (1.7) is equivalent to

$$\text{find } v \in X : \langle B\, v,\, w \rangle \;=\; \langle g,\, w \rangle \qquad \forall w \in X.$$

For a finite-dimensional subspace $X_h \subset X$, we apply the Galerkin principle:

$$\text{find } v_h \in X_h : \langle B\, v_h,\, w_h \rangle \;=\; \langle g,\, w_h \rangle \qquad \forall w_h \in X_h.$$

The choice of X_h depends on the BIE under consideration and on the unknown u (which could, e.g., be a Dirichlet trace $\gamma_0 u$ or a Neumann trace $t = \partial u / \partial n$). If the unknown is a Neumann trace, one can use a piecewise constant ansatz:

$$v_h \;=\; \sum_{j=1}^{n} v_j \, \chi_{\tau_j} \,.$$

Again, this leads to a linear system of equations for the coefficients $[v_j]_{j=1}^n$.

Collocation BEM	Galerkin BEM
Easier to implement: evaluation of single integral assembling faster	Harder to implement: evaluation of double integral assembling costy
In general, non-symmetric system matrix, even if B is self-adjoint.	If B is self-adjoint, then the system matrix is symmetric.
Stability and convergence on polyhedral surfaces still an open question, in particular for first kind BIEs.	Stability and convergence can be analyzed similarly to FEM, e.g., using Céa's lemma and interpolation error estimates.

For reasons above, we will focus on the Galerkin BEM in this lecture. A priori error estimates for the Dirichlet and Neumann problem are to be derived.

BEM vs. FEM A direct comparison of FEM and BEM has to be done with care. Rule of thumb: if FEM works efficiently, then use it. However, there are situations where the construction of a "volume" mesh is difficult, e.g., for exterior problems.

■ Further Topics (not all covered in this lecture)

More General PDEs The major part of the lecture is devoted to the Poisson equation,

$$-\Delta u \;=\; f.$$

Most of the results can be generalized to the second order PDE

$$-\mathrm{div}(A\nabla u) + \vec{b}\cdot\nabla u + c\,u \;=\; f,$$

with constant coefficients $A \in \mathbb{R}^{d\times d}$ ($A > 0$), $\vec{b} \in \mathbb{R}^d$, and $c \in \mathbb{R}$, but this introduces more technicalities. Special case: the *Helmholtz equation* $\Delta u + \kappa^2 u = 0$. The exterior Helmholtz problem (modelling acoustic waves) is very often solved by BEM in practice. Another prominent application of BEM are electromagnetic waves (governed by Maxwell's equations).

Calculation of Matrix Entries The entries of the Galerkin BEM matrices are double integrals with singular kernels. In rare cases, explicit formulae can be found. In general, one has to approximate either one or both integrals by quadrature.

Fast BEM Of course, we want to assemble and solve the BEM systems efficiently. But as will turn out, the BEM system matrices are dense, which causes trouble in particular for three-dimensional problems. However, the matrices can be approximated by *data-sparse matrices*, or at least the matrix-vector product can be realized in quasi-optimal complexity. Such techniques are summarized under the keyword *fast BEM*.

Conditioning of System Matrices An efficient realization of the matrix-vector product makes the system amenable to iterative solvers. Therefore, one has to study the condition number.

BEM-FEM Coupling In a direct comparision of BEM and FEM, there are pros and cons for either of the two methods. In some situations, one wants to exploit the advantages of both "worlds" (*mariage à la mode*): use FEM in one part of the domain and BEM in the other part. On the interface, one needs to couple correctly.

Chapter 2

Variational Framework

Before we come to boundary integral equations, we need to fix the starting point. This is the weak formulation of an elliptic 2nd order PDE in a bounded domain as well as in its exterior. Therefore, we need Sobolev spaces (on domains and boundaries) and trace operators. We briefly discuss solvability of certain boundary value problems.

2.1 Sobolev Spaces

2.1.1 Lipschitz Domains

Throughout the whole lecture (and in contrast to many textbooks), by a *domain* Ω, we understand an *open and connected* subset of $\mathbb{R}^d$, where $d = 2$ or 3. Note that our definition does not include *boundedness*. The *boundary* is denoted by

$$\Gamma := \partial\Omega = \overline{\Omega} \cap (\mathbb{R}^d \setminus \Omega).$$

We discuss smoothness assumptions on the domain Ω.

Definition 2.1 (Lipschitz hypograph)**.** Let Ω be of the form

$$\Omega = \{x \in \mathbb{R}^d : x_d < \zeta(x_1, \ldots, x_{d-1})\}$$

with a Lipschitz continuous function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$, i.e.,

$$\exists M = \text{const} : \quad |\zeta(x') - \zeta(y)| \leq M\,|x' - y'| \qquad \forall x',\, y' \in \mathbb{R}^{d-1}.$$

Then Ω is called *Lipschitz hypograph*.

Apparently, the boundary of a Lipschitz hypograph is parametrized by

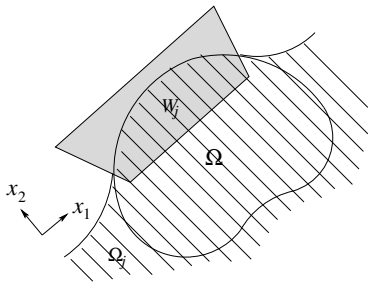
$$\Gamma = \{(x', \zeta(x')) : x' \in \mathbb{R}^{d-1}\}.$$

Definition 2.2 (Lipschitz domain)**.** A domain Ω is called a *Lipschitz domain* if its boundary Γ is compact and if there exist finite families $\{W_j\}_{j=1}^n$ and $\{\Omega_j\}_{j=1}^n$ such that

(i) $\{W_i\}_{j=1}^n$ is a finite open cover of Γ, i. e., $W_j \subset \mathbb{R}^d$ are open sets and $\Gamma \subset \bigcup_{j=1}^n W_j$,

(ii) each Ω_j can be transformed to a Lipschitz hypograph (with function ζ_j) by rotation plus translation,

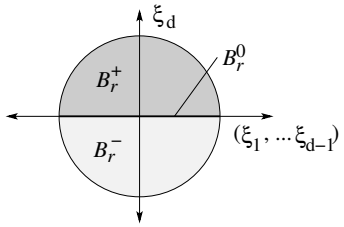
(iii) $W_j \cap \Omega = W_j \cap \Omega_j$ for each $j = 1, \ldots, n$.



We shall give a second definition of Lipschitz domains (cf. [Sauter/Schwab, Def. 2.2.7], [Grisvard, Def. 1.2.1.2]), which is slightly weaker than Definition 2.2 and thus called *weakly Lipschitz domain*.

Let $B_r := \{\xi \in \mathbb{R}^d : |\xi| < r\}$ be the open ball of radius r around the origin and define

$$\begin{aligned} B_r^+ &:= \{\xi \in B_r : \xi_d > 0\} \\ B_r^0 &:= \{\xi \in B_r : \xi_d = 0\} \\ B_r^- &:= \{\xi \in B_r : \xi_d < 0\} \end{aligned}$$



where ξ_d is the d-th coordinate.

Definition 2.3 (weakly Lipschitz)**.** We call a domain Ω *weakly Lipschitz* if its boundary Γ is compact and if there exists a finite open cover $\{U_j\}_{j=1}^n$ of Γ and bijective functions $\chi_j : \overline{B}_2 \to \overline{U}_j$ such that

(i) χ_j and χ_j^{-1} are Lipschitz continuous,
(ii) $\chi_j(B_2^+) = U_j \cap \Omega$,
(iii) $\chi_j(B_2^0) = U_j \cap \Gamma$,
(iv) $\chi_j(B_2^-) = U_j \cap (\mathbb{R}^d \setminus \Omega)$.

In this definition, B_2 may be replaced a suitable bounded open set $B \subset \mathbb{R}^d$ that can divided similarly into subsets B^+, B^0, B^- with $\xi_d > 0$, $\xi_d = 0$, $\xi_d < 0$, respectively.

Note that $\mathbb{R}^d$ is (weakly) Lipschitz ($\Gamma = \emptyset$), and that a domain Ω is (weakly) Lipschitz if and only if $\mathbb{R}^d \setminus \overline{\Omega}$ is (weakly) Lipschitz.

Many polygons and polyhedra are Lipschitz domains. Some counterexamples are shown in Figure 2.1: in (a) the graph fails to be Lipschitz, in (b) the boundary fails to be on one side, and in (c) the boundary fails to be a graph at all. However, domain (c) is weakly Lipschitz.

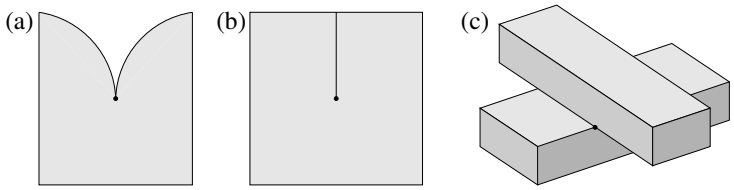


Figure 2.1: (a)–(c) are not Lipschitz domains, (c) is weakly Lipschitz.

Lemma 2.4. *A Lipschitz domain is weakly Lipschitz.*

Exercise 1. Prove Lemma 2.4 by providing U_j, χ_j in terms of W_j, Ω_j, ζ_j. You don't have to carry out geometric issues in detail.

Lemma 2.5 (Rademacher)**.** *A weakly Lipschitz domain Ω has a* surface measure *s on Γ and an* outward unit normal vector *n that exists s-almost everywhere on Γ, with $n \in L^\infty(\Gamma)$.*

The proof uses the fact that a Lipschitz continuous function is Fréchet differentiable almost everywhere and its gradient is in L^∞.

Exercise 2. Show that for a *Lipschitz hypograph* $\Omega \subset \mathbb{R}^2$, $\mathbb{R}^3$, the surface measures are $ds = \sqrt{1 + |\zeta'(x_1)|^2}\, dx_1$ and $ds = \sqrt{1 + |\nabla\zeta(x_1, x_2)|^2}\, d(x_1, x_2)$, respectively. In both cases, calculate the unit normal vector n.

We will mostly use the (weaker) Definition 2.3 of Lipschitz domains. Sometimes, we will require higher smoothness of the boundary:

Definition 2.6. We call a domain Ω to be of

- *class* C^k (for $k \geq 1$) if Ω is weakly Lipschitz and if the functions χ_j from Def. 2.3 satisfy $\chi_j \in C^k$ and $\chi_j^{-1} \in C^k$,
- *class* $C^{k,1}$ (for $k \geq 0$) if Ω is weakly Lipschitz and χ_j, χ_j^{-1} are in $C^{k,1}$, i.e., k-times differentiable and the k-th derivatives Lipschitz continuous,

2.1.2 Distributions

Recall that

$$C_0^\infty(\Omega) := \{u \in C^\infty(\Omega) : \mathrm{supp}(u) \subset\subset \Omega\}$$

denotes the C^∞ functions with compact support in Ω, where $\mathrm{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}$ for $u \in C(\Omega)$.

Let $L^1_{\text{loc}}(\Omega)$ denote the set of measurable functions $u : \Omega \to \mathbb{R}$ that are integrable over every compact subset of Ω. One can show that if for two functions $u_1, u_2 \in L^1_{\text{loc}}(\Omega)$,

$$\int_\Omega u_1 \, \phi \, dx \;=\; \int_\Omega u_2 \, \phi \, dx \qquad \forall \phi \in C_0^\infty(\Omega), \tag{2.1}$$

then $u_1 = u_2$ almost everywhere on Ω.

Schwartz Distributions

Definition 2.7. $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$. We say that a sequence ϕ_n in $\mathcal{D}(\Omega)$ *converges sequentially* to $\phi \in \mathcal{D}(\Omega)$; we write

$$\phi_n \;\to_{\mathcal{D}(\Omega)}\; \phi,$$

if for all compact subsets $K \subset \Omega$ and for all multi-indices α,

$$\partial^\alpha \phi_n \to \partial^\alpha \phi \quad \text{uniformly in } K.$$

Definition 2.8. The set of *Schwartz distributions* is given by

$$\mathcal{D}^*(\Omega) \;:=\; \{\psi : \mathcal{D}(\Omega) \to \mathbb{R} : \text{linear and continuous}\},$$

where ψ is said to be continuous if

$$\phi_n \to_{\mathcal{D}(\Omega)} \phi \qquad \Longrightarrow \qquad \psi(\phi_n) \to \psi(\phi).$$

For the evaluation of a distribution at a function, we write

$$\langle \psi, \, \phi \rangle_\Omega \;:=\; \psi(\phi) \qquad \text{for } \phi \in \mathcal{D}(\Omega), \; \psi \in \mathcal{D}^*(\Omega).$$

If the domain Ω is clear from context, we omit the subscript Ω.

Example 2.9. For $y \in \Omega$, the *Dirac delta distribution* δ_y is given by

$$\langle \delta_y, \, \phi \rangle \;:=\; \phi(y) \qquad \text{for } \phi \in \mathcal{D}(\Omega).$$

Definition 2.10. Any $f \in L^1_{\text{loc}}(\Omega)$ induces a distribution $\bar{f} \in \mathcal{D}^*(\Omega)$:

$$\langle \bar{f}, \, \phi \rangle \;:=\; \int_\Omega f \, \phi \, dx \qquad \text{for } \phi \in \mathcal{D}(\Omega).$$

If a distribution is induced by a *function* f as above, we call it *regular*. The Dirac delta distribution is not regular. One can show that the linear map $f \mapsto \bar{f}$ is one-to-one. Hence, we may *identify* locally integrable functions with their corresponding distributions.

Definition 2.11. The *distributional derivative* $\partial^\alpha \psi \in \mathcal{D}^*(\Omega)$ of a distribution $\psi \in \mathcal{D}^*(\Omega)$ is defined by

$$\langle \partial^\alpha \psi, \, \phi \rangle \;=\; (-1)^{|\alpha|} \langle \psi, \, \partial^\alpha \phi \rangle \qquad \text{for } \psi \in \mathcal{D}(\Omega).$$

(The distributional derivative of a distribution is again a distribution.)

More Distributions

Definition 2.12. $\mathcal{E}(\Omega) := C^\infty(\Omega)$. We define sequential convergence (in symbols: $\phi_n \to_{\mathcal{E}(\Omega)} \phi$) and continuity analogously to Def. 2.7 and set

$$\mathcal{E}^*(\Omega) := \{\psi : \mathcal{E}(\Omega) \to \mathbb{R} : \text{linear and continuous}\}.$$

Apparently,

$$\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega), \qquad \mathcal{E}^*(\Omega) \subset \mathcal{D}^*(\Omega).$$

Definition 2.13 (tempered distributions)**.** We define the Schwartz space of *rapidly decreasing functions*

$$\mathcal{S}(\mathbb{R}^d) := \left\{\varphi \in C^\infty(\Omega) : \sup_{x\in\mathbb{R}^d} |x^\alpha\,(\partial^\beta\varphi)(x)| < \infty \quad \forall \text{ multi-indices } \alpha,\, \beta\right\},$$

where $x^\alpha := x_1^{\alpha_1}\, x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$. We equip $\mathcal{S}(\mathbb{R}^d)$ with the following sequential convergence,

$$\phi_n \to_{\mathcal{S}(\mathbb{R}^d)} \phi \quad \iff \quad \forall \alpha, \beta : x^\alpha(\partial^\beta \phi_n) \to x^\alpha(\partial^\beta\phi) \text{ uniformly in } \mathbb{R}^d.$$

With this convergence, we can define continuity and define the space of *temperate distributions*

$$\mathcal{S}^*(\mathbb{R}^d) := \{\psi : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R} : \text{linear and continuous}\}.$$

Apparently,

$$\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset \mathcal{E}(\mathbb{R}^d), \qquad \mathcal{E}^*(\mathbb{R}^d) \subset \mathcal{S}^*(\mathbb{R}^d) \subset \mathcal{D}^*(\mathbb{R}^d).$$

Fourier Transform

For $u \in L^1(\mathbb{R}^d)$ we define its *Fourier transform*[1] $\mathcal{F}u$ by

$$(\mathcal{F}u)(\xi) := \widehat{u}(\xi) := \int_{\mathbb{R}^d} e^{-i2\pi\xi\cdot x} u(x)\, dx \qquad \text{for } \xi \in \mathbb{R}^d.$$

The adjoint $\mathcal{F}^*$ operator is defined by replacing $-i$ above by i. One can show that if $u \in C(\mathbb{R}^d)$ and $u,\, \widehat{u} \in L^1(\mathbb{R}^d)$, then $\mathcal{F}^*\mathcal{F}u = u = \mathcal{F}\mathcal{F}^*u$. Elementary calculations show that for $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$(\mathcal{F}\partial^\alpha\phi)(\xi) = (i2\pi\xi)^\alpha\, \hat{\phi}(\xi), \qquad (\mathcal{F}^*(-i2\pi x)^\alpha\phi(x))(\xi) = \partial^\alpha\hat{\phi}(\xi).$$

Consequently, $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a (sequentially) continuous operator and $\mathcal{F}^* : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is its (sequentially) continuous inverse.

[1] For a precise definition, one must actually use spaces $L^p(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, etc. with values in $\mathbb{C}$ rather than $\mathbb{R}$.

Plancherel's Theorem* The formulae $\langle \mathcal{F}u, v\rangle = \langle u, \mathcal{F}u\rangle$ and $\langle \mathcal{F}^* u, v\rangle = \langle u, \mathcal{F}^* u\rangle$ are valid for u, $v \in \mathcal{S}(\mathbb{R}^d)$, and they are used to define (consistent) extensions

$$\mathcal{F} : \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^d), \qquad \mathcal{F}^* : \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^d).$$

Plancherel's Theorem states that

$$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad \mathcal{F}^* : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),$$

and that these maps are inverse to each other. Furthermore,

$$(\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^d)} = (u, v)_{L^2(\mathbb{R}^d)} = (\mathcal{F}^* u, \mathcal{F}^* v)_{L^2(\mathbb{R}^d)} \qquad \text{for } u,\, v \in L^2(\mathbb{R}^d),$$

and consequently, $\|\mathcal{F}u\|_{L^2(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}^* u\|_{L^2(\mathbb{R}^d)}$.

2.1.3 Sobolev Spaces on a Domain

First Definition

Recall that $L^2(\Omega) = \{v : \Omega \to \mathbb{R} \text{ measurable} : \|v\|_{L^2(\Omega)} < \infty\}$, where

$$\|v\|_{L^2(\Omega)} := \Big(\int_\Omega |v|^2 \, dx \Big)^{1/2}.$$

Since $L^2(\Omega) \subset L^1_{\text{loc}}$, such functions are (regular) distributions.

Definition 2.14. (i) For any $k \in \mathbb{N}_0$ we define[2]

$$H^k(\Omega) := \{v \in L^2(\Omega) : \partial^\alpha v \in L^2(\Omega) \quad \forall \text{ multi-indices } \alpha,\ |\alpha| \le k\},$$

equipped with the inner product

$$(v,\, w)_{H^k(\Omega)} := \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha v \, \partial^\alpha w \, dx$$

and the induced norm $\|v\|_{H^k(\Omega)} := (\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha v| \, dx)^{1/2}$.

(ii) For $s \in \mathbb{R}^+$ of the form $s = k + \sigma$ with $k \in \mathbb{N}_0$, $\sigma \in (0, 1)$, we define

$$H^s(\Omega) := \{v \in H^k(\Omega) : |v|_{H^s(\Omega)} < \infty\},$$

where

$$|v|_{H^s(\Omega)} := \Big(\sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{d+2\sigma}} \, dx \, dy \Big)^{1/2},$$

equipped with the norm $\|v\|_{H^s(\Omega)} := (|v\|^2_{H^k(\Omega)} + |v|^2_{H^s(\Omega)})^{1/2}$ (which has an associated inner product).

[2]The spaces as defined in (i), (ii) are usually denoted by $W^{k,2}(\Omega)$, $W^{s,2}(\Omega)$. Denoting them by $H^s(\Omega)$ is justified by Def. 2.17 and Thm. 2.18 below.

Lemma 2.15. *For $s \in \mathbb{R}_0^+$, the space $H^s(\Omega)$ is a Hilbert space. $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$ with respect to the $\|\cdot\|_{H^s(\Omega)}$-norm.*

Definition 2.16. For $s \in \mathbb{R}^+$, we set

$$H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}} \quad \text{(closure of } C_0^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^s(\Omega)}).$$

Equipped with $\|\cdot\|_{H^s(\Omega)}$, this is a Hilbert space. $H_0^0(\Omega) = H^0(\Omega) = L^2(\Omega)$.

Note that the above definitions include the case $\Omega = \mathbb{R}^d$ (they actually require only that Ω is a non-empty open set).

Second Definition

For $s \in \mathbb{R}$, we define the *Bessel potential* $\mathcal{J}^s : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{J}^s u(x) := \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s/2}\, \widehat{u}(\xi)\, e^{i2\pi\, \xi\cdot x}\, d\xi \qquad \text{for } x \in \mathbb{R}^d,$$

where $\widehat{u} := \mathcal{F}u \in \mathcal{S}(\mathbb{R}^d)$ denotes the Fourier transform of $u \in \mathcal{S}(\mathbb{R}^d)$. Hence,

$$(\mathcal{F}\mathcal{J}^s u)(\xi) = (1 + |\xi|^2)^{s/2}\, \widehat{u}(\xi),$$

and so $\mathcal{J}^s$ can be thought of a kind of differential operator of order s. We extend the Bessel potential to $\mathcal{J}^s : \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^d)$ by

$$\langle \mathcal{J}^s \psi,\, \phi\rangle := \langle \psi,\, \mathcal{J}^s \phi\rangle \qquad \text{for } \psi \in \mathcal{S}^*(\mathbb{R}^d),\, \phi \in \mathcal{S}(\mathbb{R}^d).$$

Exercise 3. Show that $\langle \mathcal{J}^s\psi,\, \phi\rangle = \langle \psi,\, \mathcal{J}^s\phi\rangle$ holds for $\phi,\, \psi \in \mathcal{S}(\mathbb{R}^d)$ such the extension is indeed justified.

Definition 2.17 (alternative (usual) definition of H^s)**.** For $s \in \mathbb{R}$, we define

$$H^s(\mathbb{R}^d) := \{u \in \mathcal{S}^*(\mathbb{R}^d) : \mathcal{J}^s u \in L^2(\mathbb{R}^d)\},$$

with inner product $(\mathcal{J}^s u,\, \mathcal{J}^s v)_{L^2(\mathbb{R}^d)}$. For an open set $\Omega \subset \mathbb{R}^d$, we define

$$H^s(\Omega) := \{u \in \mathcal{D}^*(\Omega) : u = v_{|\Omega} \text{ for some } v \in H^s(\mathbb{R}^d)\},$$

equipped with norm $\|u\|_{H^s(\Omega)} := \inf_{v \in H^s(\mathbb{R}^d),\, u = v_{|\Omega}} \|v\|_{H^s(\mathbb{R}^d)}$. Both spaces make into Hilbert spaces.

Theorem 2.18. *If Ω is (weakly) Lipschitz, then Def. 2.14 and Def. 2.17 of $H^s(\Omega)$ are equivalent for $s \in \mathbb{R}_0^+$, i.e., the two definitions lead to identical sets of functions and the norms are equivalent.*

Dual Spaces

For $s \in \mathbb{R}_0^+$, let

$$H^s(\Omega)^*, \qquad H_0^s(\Omega)^*$$

denote the respective duals with the usual dual norm. In the literature, $H_0(\Omega)^*$ is sometimes—but not always—denoted by $H^{-s}(\Omega)$.

Recall that we identify functions in $L^1_{\mathrm{loc}}(\Omega)$ with their associated distributions. In that sense,

$$\mathcal{D}(\Omega) \subset H_0^s(\Omega) \subset H^s(\Omega) \subset L^2(\Omega) \subset H^s(\Omega)^* \subset H_0^s(\Omega)^* \subset \mathcal{D}^*(\Omega).$$

2.1.4 Sobolev Spaces on the Boundary

Let Ω be a weakly Lipschitz domain. Because of Lemma 2.5, we have a surface measure s and can therefore define

$$L^2(\Gamma) := \big\{v : \Gamma \to \mathbb{R} : s\text{-measurable and } \int_\Gamma |v|^2 \, ds < \infty\big\}.$$

The definition of Sobolev spaces on Γ is more technical than on Ω.

Recall the finite open cover $\{U_j\}_{j=1}^n$ and the associated bi-Lipschitz functions $\chi_j : \overline{B}_2 \to \overline{U}_j$. We now define the restrictions

$$\chi_j^\star : B_2^0 \to U_j \cap \Gamma : (\xi_1, \dots, \xi_{d-1}) \mapsto \chi_j(\xi_1, \dots, \xi_{d-1}, 0)$$

(here we treat B_2^0 as a subset of $\mathbb{R}^{d-1}$). Let $\{\beta_j\}_{j=1}^n$ be a *partition of unity subordinate to* $\{U_j \cap \Gamma\}_{j=1}^n$:

$$\beta_j : \Gamma \to [0,\, 1], \qquad \sum_{j=1}^n \beta_j = 1 \text{ on } \Gamma, \qquad \operatorname{supp}(\beta_j) \subset U_j \cap \Gamma.$$

One can show that such a partition always exists, even with $\beta_j \in C^\infty$, see [McLean, Cor. 3.22]). Therefore, we can achieve that $\beta_j \circ \xi_j^\star$ is Lipschitz continuous on $\overline{B}_2^0$ and has support compact in B_2^0. Functions $u : \Gamma \to \mathbb{R}$ can be localized using this partition of unity:

$$\operatorname{supp}(\beta_j u) \subset U_j \cap \Gamma.$$

We now define the *pullback* of such a localized function:

$$(\beta_j u) \circ \chi_j^\star : B_2^0 \to \mathbb{R}.$$

We use this pullback to define smoothness of the original function $u : \Gamma \to \mathbb{R}$.

Definition 2.19. For a $C^{k-1,1}$ domain Ω (i.e. weakly Lipschitz if $k = 1$) and for $s \in [0,\, k]$, we define

$$H^s(\Gamma) \;:=\; \{u \in L^2(\Gamma) : (\beta_j u) \circ \chi_j^\star \in H^s(B_2^0) \quad \forall j\}.$$

One can show that the above definition is *invariant* of the actual choice of the "coordinate system" U_j, χ_j, β_j. This invariance property is lost if $s > k$.

A natural norm would be $\big(\sum_{j=1}^n \|(\beta_j u) \circ \chi_j^\star\|^2_{H^s(B_2^0)}\big)^{1/2}$, but this norm *depends* on the choice of the coordinate system. The definition below introduces an *invariant* norm.

Definition 2.20. For a $C^{k-1,1}$ domain Ω with boundary Γ and for $s \in [0,\, k]$, the *Sobolev-Slobodeckij norm* $\|v\|_{H^s(\Gamma)}$ is given as follows.

(i) For $k \in \mathbb{N}_0$,

$$\|v\|^2_{H^k(\Gamma)} \;=\; \sum_{|\alpha| \le k} \|v_\alpha\|^2_{L^2(\Gamma)},$$

where $v_\alpha : \Gamma \to \mathbb{R}$ are defined using differentiation in B_0^2:

$$v_\alpha(x) \;=\; \sum_{j=1}^n \partial_\xi^\alpha((\beta_j v) \circ \chi_j^\star)(\xi), \qquad \text{for } \xi = (\chi_j^\star)^{-1}(x).$$

If $x \notin U_j \cap \Gamma$ for some j, then the corresponding term in the sum above is skipped.

(ii) For $s = k + \sigma$, with $k \in \mathbb{N}_0$ and $\sigma \in (0, 1)$,

$$\|v\|^2_{H^s(\Gamma)} \;=\; \sum_{|\alpha| \le k} \Big(\|v_\alpha\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|v_\alpha(x) - v_\alpha(y)|^2}{|x - y|^{d-1+2\sigma}}\, ds_x\, ds_y,$$

The spaces $H^s(\Gamma)$ with the above norm are Hilbert spaces.
Important special case $s = 1/2$:

$$\|v\|^2_{H^{1/2}(\Gamma)} = \|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{1/2}(\Gamma)}, \quad |v|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x - y|^d}\, ds_x\, ds_y\,,$$

and $H^{1/2}(\Gamma) = \{v \in L^2(\Gamma) : \|v\|_{H^{1/2}(\Gamma)} < \infty\}$ (intrinsic norm).

Definition 2.21. For Ω, Γ, and s as in Definition 2.19,

$$H^{-s}(\Gamma) := H^s(\Gamma)^*$$

equipped with the usual dual norm.

2.1.5 Sobolev Spaces on a Manifold

Definition 2.22. Let $\Gamma_1 \subset \Gamma$ be an open submanifold of the boundary Γ of a weakly Lipschitz (or $C^{k-1,1}$ domain), and let $s \in [0,1]$ (or $[0,k]$, resp.).

(i) $H^s(\Gamma_1) := \{v_{|\Gamma_1} : v \in H^s(\Gamma)\}$, equipped with the norm $\|v\|_{H^s(\Gamma_1)}$ defined analogously to Def. 2.20, replacing Γ by Γ_1.

(ii) $H^s_0(\Gamma_1) := \overline{\mathcal{D}(\Gamma_1)}^{\|\cdot\|_{H^s(\Gamma_1)}}$,
where $\mathcal{D}(\Gamma_1) := \{v_{|\Gamma} : v \in \mathcal{D}(\mathbb{R}^d),\ \mathrm{supp}(v_{|\Gamma}) \subset \overline{\Gamma}_1\}$.

For $s = \lfloor s \rfloor + \sigma$ and $\sigma \neq 1/2$, the extension by zero of a function $v \in H^s_0(\Gamma_1)$ from Γ_1 to Γ belongs to $H^s(\Gamma)$, but in general not for $\sigma = 1/2$!

(iii) $\widetilde{H}^s(\Gamma_1) := \overline{\mathcal{D}(\Gamma_1)}^{\|\cdot\|_{H^s(\Gamma)}} = \{v_{|\Gamma_1} : v \in H^s(\Gamma),\ \mathrm{supp}(v) \subset \overline{\Gamma}_1\}$.

The extension by zero of a function from $\widetilde{H}^s(\Gamma_1)$ to Γ always lies in $H^s(\Gamma)$. Equipped with the norms

$$\|u\|_{H^s(\Gamma_1)} := \inf_{\substack{v \in H^s(\Gamma) \\ v_{|\Gamma_1} = u}} \|v\|_{H^s(\Gamma)}, \qquad \|u\|_{\widetilde{H}^s(\Gamma_1)} := \|\widetilde{u}\|_{H^s(\Gamma)}, \tag{2.2}$$

where $\widetilde{u}$ denotes the extension by zero, $H^s(\Gamma)$, $\widetilde{H}^s(\Gamma)$ are Hilbert spaces. For $\sigma < 1/2$, $\widetilde{H}^s(\Gamma_1) = H^s_0(\Gamma_1) = H^s(\Gamma_1)$. For $\sigma > 1/2$, $\widetilde{H}^s(\Gamma_1) = H^s_0(\Gamma_1)$.

Remark 2.23. If Γ_1 is "Lipschitz relatively to Γ", then intrinsic norms for $H^s(\Gamma_1)$, $\widetilde{H}^{1/2}(\Gamma_1)$ (equivalent to the norms in (2.2)) are given by

$$\Big(\int_{\Gamma_1} \int_{\Gamma_1} \frac{|u(x) - u(y)|^2}{|x-y|^{d-1+2s}}\, ds_x\, ds_y \Big)^{1/2},$$
$$\Big(\int_{\Gamma_1} \int_{\Gamma_1} \frac{|u(x) - u(y)|^2}{|x-y|^{d}}\, ds_x\, ds_y + \int_{\Gamma_1} \frac{|u(x)|^2}{\mathrm{dist}(x,\, \partial\Gamma_1)}\, ds_x \Big)^{1/2},$$

respectively, for $s < 1$ (and analogously defined for $s > 1$).

Definition 2.24. Let $\Gamma_1 \subset \Gamma$ be as in Definition 2.22 and set

$$H^{-s}(\Gamma_1) := \widetilde{H}^s(\Gamma_1)^*, \qquad \widetilde{H}^{-s}(\Gamma_1) := H^s(\Gamma_1)^*$$

(this definitinion is conistent with Def. 2.21, because $\widetilde{H}^s(\Gamma) = H^s(\Gamma)$ for the *closed* boundary Γ).

2.1.6 The Trace Operator

Theorem 2.25 (trace theorem)**.** *We define the trace operator*

$$\gamma_0 : C^\infty(\overline{\Omega}) \to C^\infty(\Gamma) : u \mapsto u_{|\Gamma}\,.$$

If $1/2 < s < k$ *and if* Ω *is a* $C^{k-1,1}$*-domain, then* γ_0 *has a unique extension to a bounded linear operator*

$$\gamma_0 : H^s(\Omega) \to H^{s-1/2}(\Gamma).$$

In particular, there exists a constant $C_{\rm tr}$ *(depending on* Ω *and* s*) such that*

$$\|\gamma_0 u\|_{H^{s-1/2}(\Gamma)} \;\le\; C_{\rm tr}\;\|u\|_{H^s(\Omega)} \qquad \forall u \in H^s(\Omega).$$

If Ω *is weakly Lipschitz, then* γ_0 *is bounded for* $1/2 < s < 3/2$*.*

Theorem 2.26 (inverse trace theorem)**.** *Let* Ω *and* s *be as in Thm. 2.25. Then the trace operator* γ_0 *has a bounded right inverse*

$$\mathcal{E} : H^{s-1/2}(\Gamma) \to H^s(\Omega) \qquad \textit{with} \qquad \gamma_0\mathcal{E}w = w \quad \forall w \in H^{s-1/2}(\Gamma).$$

In particular, there exists a constant $C_{\rm IT}$ *(depending on* Ω *and* s*) such that*

$$\|\mathcal{E}w\|_{H^s(\Omega)} \;\le\; C_{\rm IT}\;\|w\|_{H^{s-1/2}(\Gamma)} \qquad \forall w \in H^{s-1/2}(\Gamma).$$

Remark 2.27.

1. Theorem 2.25 and Theorem 2.26 imply that the trace operator γ_0 maps from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$ and is *surjective*.
2. The statements of Theorem 2.25 and Theorem 2.26 hold as well if Γ is replaced by $\Gamma_1 \subset \Gamma$.
3. The trace operator is *not* continuous from $H^{1/2}(\Omega)$ to $L^2(\Gamma)$!
4. For $1/2 < s \le k$, $H^s_0(\Omega) = \{v \in H^s(\Omega) : \gamma_0(\partial^\alpha v) = 0\ \forall |\alpha| < s - 1/2\}$ (non-trivial, see e.g. [McLean, Thm. 3.40]).

Occasionally, we write $u_{|\Gamma}$ and $u_{|\Gamma_1}$ instead of $\gamma_0 u$ and $(\gamma_0 u)_{|\Gamma_1}$.

Exercise 4. (a) Show that $\|u\|_{\star,H^{1/2}(\Gamma)} := \inf_{\substack{v\in H^1(\Omega)\\ v_{|\Gamma}=u}} \|v\|_{H^1(\Omega)}$ is an equivalent norm to $\|u\|_{H^{1/2}(\Gamma)}$ (provide the constants of equivalence).

(b) Prove that the infimum in (a) is attained at a unique function. *Hint:* show that the quadratic functional $v \mapsto (v, v)_{H^1(\Omega)}$ is convex and bounded from below.

2.1.7 Compact Embedding

Theorem 2.28 (Rellich). *Let Ω be weakly Lipschitz. For $0 \le s \le t$, the inclusions*

$$H^t(\Omega) \subset H^s(\Omega), \qquad H^t(\Gamma) \subset H^s(\Gamma)$$

are compact (where in the second case, Ω must be a $C^{k-1,1}$-domain and $t \le k$).

2.1.8 Poincaré and Friedrichs Inequalities

Theorem 2.29 (Poincaré's inequality). *Let Ω be a bounded weakly Lipschitz domain. Then there exists a constant C_P (depending on Ω) such that*

$$\|u\|_{L^2(\Omega)} \le C_P\, |u|_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega),\ \int_\Omega u\, dx = 0.$$

Proof: (a) Indirect proof based on the compact embedding $H^1(\Omega) \subset L^2(\Omega)$. (b) For special domains, direct proofs are available. E.g., if Ω is convex, then $C_P = \mathrm{diam}(\Omega)/\pi$.

Remark 2.30 (other Poincaré inequalities).

1. If $\Gamma_1 \subset \Gamma$ is a submanifold of positive surface measure, then the inequality from Theorem 2.29 holds for all $u \in H^1(\Omega)$ with $\int_{\Gamma_1} u\, ds = 0$ (with a different constant).
2. For the boundary Γ of a weakly Lipschitz domain, there exists a constant C'_P (depending on Γ) such that

$$\|u\|_{L^2(\Gamma)} \le C'_P\, |u|_{H^{1/2}(\Gamma)} \qquad \forall u \in H^{1/2}(\Gamma),\ \int_\Gamma u\, ds = 0.$$

Theorem 2.31 (Poincaré-Friedrichs inequality). *Let Ω be bounded and let $\Gamma_1 \subset \Gamma$ be a submanifold of positive surface measure. Then there exists a constant C_F (depending on Ω and Γ_1) such that*

$$\|u\|_{L^2(\Omega)} \le C_F\, |u|_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega),\ u_{|\Gamma_1} = 0.$$

Remark 2.32. The classical *Friedrichs inequality* is formulated for $\Gamma_1 = \Gamma$. In that case, C_F can be expressed explicitly in terms of $\mathrm{diam}(\Omega)$. Often, the terms *Friedrichs* and *Poincaré* inequality are interchanged in the literature.

Exercise 5. Show (a) that the seminorm $|u|_{\star,H^{1/2}(\Gamma)} := \inf_{\substack{v \in H^1(\Omega) \\ v_{|\Gamma} = u}} |v|_{H^1(\Omega)}$ is equivalent to the seminorm $|u|_{H^{1/2}(\Gamma)}$, and (b) that the infimum is attained at a unique function (called the *harmonic extension of u*). *Hint:* use the trace theorems, Poincaré's inequality, and Remark 2.30.

2.2 Interior Boundary Value Problems

Throughout this section, $\Omega \subset \mathbb{R}^d$ shall denote a *bounded* domain that is weakly Lipschitz.

2.2.1 The Distributional PDE

For simplicity, we consider a linear second-order partial differential operator

$$L\,u \;=\; -\mathrm{div}(A\,\nabla u) + \cancel{b\cdot\nabla u} + c\,u, \tag{2.3}$$

with *constant* coefficients $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $b = 0$, and $c \in \mathbb{R}$, such that $\xi^\top A\,\xi > 0$, where div denotes the (distributional) divergence operator,

$$\mathrm{div}F \;:=\; \sum_{j=1}^{d} \partial_j F_j \;:=\; \sum_{j=1}^{d} \frac{\partial F_j}{\partial x_j}\,.$$

Remark* 2.33. The results of this section can be generalized to more general coefficients $A \in L^\infty(\Omega)^{d\times d}_{\mathrm{sym}}$, $b \in L^\infty(\Omega)^d$, and $c \in L^\infty(\Omega)$ such that A is *strongly elliptic*:

$$\exists \alpha_0 > 0: \quad \xi^\top A(x)\xi \geq \alpha_0\,|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d\ \forall x \in \Omega \text{ a.e.}$$

In the weak formulation, we are interested in solutions $u \in H^1(\Omega)$ of the *distributional PDE*

$$L\,u \;=\; f \qquad \text{in } \mathcal{D}^*(\Omega), \tag{2.4}$$

where $f \in H^1(\Omega)^* \subset \mathcal{D}^*(\Omega)$. Typically, even $f \in L^2(\Omega)$. Equation (2.4) is naturally associated to the bilinear form

$$a(u,\,v) \;:=\; \int_\Omega A\,\nabla u\cdot\nabla v + c\,u\,v\,dx, \tag{2.5}$$

because

$$\langle L\,u,\,\phi\rangle \;=\; a(u,\,\phi) \qquad \forall \phi \in \mathcal{D}(\Omega). \tag{2.6}$$

(Actually, for more general coefficients, (2.6) is used to *define* $L\,u \in \mathcal{D}^*(\Omega)$.) The bilinear form is H^1-*bounded*:

$$\exists \overline{a} = \text{const}: \quad |a(u,\,v)| \;\leq\; \overline{a}\,\|u\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)} \qquad \forall u,\,v \in H^1(\Omega).$$

By (2.6) and since $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$, (2.4) is *equivalent* to

$$a(u,\,v) \;=\; \langle f,\,v\rangle \qquad \forall v \in H^1_0(\Omega). \tag{2.7}$$

Apparently, the test functions in (2.7) *do not see* what's going on the boundary Γ.

2.2.2 The Conormal Derivative

For $u \in H^2(\Omega)$, we see that $A\nabla u \in H^1(\Omega)^d$ (because $A = \text{const}$) and $L\,u \in L^2(\Omega)$. We define the *conormal derivative*

$$\gamma_1 u \;:=\; \gamma_0(A\nabla u)\cdot n \qquad \in L^2(\Gamma), \tag{2.8}$$

where $n \in L^\infty(\Gamma)$ is the outward unit normal vector.

Lemma 2.34 (Green's first identity)**.**

$$a(u,\,v) \;=\; (L\,u,\,v)_{L^2(\Omega)} + (\gamma_1 u,\,\gamma_0 u)_{L^2(\Gamma)} \qquad \forall u \in H^2(\Omega),\; v \in H^1(\Omega).$$

Proof. On bounded, weakly Lipschitz domains, Gauss' theorem holds:

$$\int_\Gamma F\cdot n\,ds \;=\; \int_\Omega \mathrm{div} F\,dx \qquad \forall F \in C^1(\overline{\Omega})^d.$$

By the trace and density results, this formula also holds for all $F \in H^1(\Omega)^d$. We choose $F := v\,A\nabla u$. By Gauss' theorem and the product rule,

$$\begin{aligned}
\int_\Gamma v\,A\nabla u\cdot n\,ds \;&=\; \int_\Omega \sum_{j=1}^d \partial_j(v\,A\nabla u)\,dx \\
&=\; \int_\Omega \sum_{j=1}^d \left[\partial_j v(A\nabla u) + v\,\partial_j(A\nabla u)\right] dx \\
&=\; \int_\Omega \nabla v\cdot(A\nabla u)\,dx + \int_\Omega v\,\mathrm{div}(A\nabla u)\,dx.
\end{aligned}$$

The results follows now from the definitions of $a(\cdot,\cdot)$, L, γ_0, and γ_1. □

Remark* 2.35. With the same techniques, one shows that

$$a(u,\,v) \;=\; (u,\,L\,v)_{L^2(\Omega)} + (\gamma_0 u,\,\gamma_1 v)_{L^2(\Gamma)} \qquad \forall u \in H^1(\Omega),\; v \in H^2(\Omega).$$

If $b \neq 0$, the above formula has to be modified, replacing $L\,v$ by $L^*v := -\mathrm{div}(A\nabla u) - \mathrm{div}(b\,v) + c\,v$ and $\gamma_1 v$ by $\widetilde{\gamma}_1 v := \gamma_0(A\nabla v)\cdot n + (b\cdot n)\gamma_0 v$.

Unfortunately, the condition $u \in H^2(\Omega)$ does not hold in general for solutions of (2.4). The next two theorems help to define a *weak* conormal deriative.

Theorem 2.36. *For $u \in H^1(\Omega)$ with $\mathrm{div}(A\nabla u) \in L^2(\Omega)$, there exists a unique linear functional $g \in H^{-1/2}(\Gamma)$ such that*

$$a(u,\,v) \;=\; (L\,u,\,v)_{L^2(\Omega)} + \langle g,\,\gamma_0 v\rangle \qquad \forall v \in H^1(\Omega).$$

For $u \in H^2(\Omega)$, $g = \gamma_1 u$. Furthermore, there exists a constant C such that

$$\|g\|_{H^{-1/2}(\Gamma)} \;\le\; C\left(\|u\|_{H^1(\Omega)} + \|\mathrm{div}(A\nabla u)\|_{L^2(\Omega)}\right).$$

Proof. (1) We define $g \in H^{-1/2}(\Gamma) = H^{1/2}(\Gamma)^*$ by

$$\langle g,\, \phi\rangle \;:=\; a(u,\, \mathcal{E}\phi) - (L\,u,\, \mathcal{E}\phi)_{L^2(\Omega)} \qquad \text{for } \phi \in H^{1/2}(\Gamma),$$

where $\mathcal{E}$ is the right inverse of γ_0 (see Theorem 2.26).
(2) We show the identity in the lemma. Given $v \in H^1(\Omega)$ arbitrary but fixed, consider the function

$$v_0 \;:=\; v - \mathcal{E}\gamma_0 v.$$

Since $\gamma_0\mathcal{E}\gamma_0 v = \gamma_0 v$, we have $\gamma_0 v_0 = 0$ and so $v_0 \in H^1_0(\Omega)$. From the density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega)$ and the definition of the distributional divergence, it follows that

$$a(u,\, v_0) \;=\; (L\,u,\, v_0)_{L^2(\Omega)}\,.$$

By that identity, the definition of g, and because $v = v_0 + \mathcal{E}\gamma_0 v$,

$$\begin{aligned} a(u,\, v) \;&=\; a(u,\, v_0) + a(u,\, \mathcal{E}\gamma_0 v) \\ &=\; \underbrace{(L\,u,\, v_0)_{L^2(\Omega)} + (L\,u,\, \mathcal{E}\gamma_0 v)_{L^2(\Omega)}}_{=(L\,u,\,v)_{L^2(\Omega)}} + \langle g,\, \gamma_0 v\rangle. \end{aligned}$$

(3) We show boundedness of g: for (arbitrary) $\phi \in H^{1/2}(\Gamma)$,

$$\begin{aligned} |\langle g,\, \phi\rangle| \;&\le\; |a(u,\, \mathcal{E}\phi) + (L\,u,\, \mathcal{E}\phi)_{L^2(\Omega)}| \\ &=\; \Big|\int_\Omega A\,\nabla u\cdot\nabla(\mathcal{E}\phi) + \operatorname{div}(A\,\nabla u)\,\mathcal{E}\phi\,dx\Big| \\ &\le\; \overline{a}\,\|u\|_{H^1(\Omega)}\,\|\mathcal{E}\phi\|_{H^1(\Omega)} + \|\operatorname{div}(A\,\nabla u)\|_{L^2(\Omega)}\,\|\mathcal{E}\phi\|_{L^2(\Omega)} \\ &\le\; \max(\overline{a},\,1)\Big(\|u\|_{H^1(\Omega)} + \|\operatorname{div}(A\,\nabla u)\|_{L^2(\Omega)}\Big)\,\|\mathcal{E}\phi\|_{H^1(\Omega)}\,. \end{aligned}$$

Due to the inverse trace theorem, $\|\mathcal{E}\phi\|_{H^1(\Omega)} \le C_{\mathrm{IT}}\|\phi\|_{H^{1/2}(\Gamma)}$. By the definition of the dual norm,

$$\underbrace{\sup_{\phi\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{|\langle g,\, \phi\rangle|}{\|\phi\|_{H^{1/2}(\Gamma)}}}_{=:\|g\|_{H^{-1/2}(\Gamma)}} \;\le\; \max(\overline{a},\,1)C_{\mathrm{IT}}\Big(\|u\|_{H^1(\Omega)} + \|\operatorname{div}(A\,\nabla u)\|_{L^2(\Omega)}\Big).$$

(4) We show that g is unique. Suppose that both g_1 and g_2 satisfy the conclusions of the lemma. Then the difference $g_2 - g_1 \in H^{-1/2}(\Gamma)$ satisfies

$$\langle g_2 - g_1,\, \gamma_0 v\rangle \;=\; 0 \qquad \forall v \in H^1(\Omega).$$

Since the trace operator γ_0 is surjective onto $H^{1/2}(\Gamma)$, this implies that

$$\langle g_2 - g_1,\, \phi\rangle \;=\; 0 \qquad \forall \phi \in H^{1/2}(\Gamma),$$

and so $g_2 = g_1$ in the sense of $H^{-1/2}(\Gamma)$. This also shows that the construction of g is actually independent of the particular choice of $\mathcal{E}$. □

Definition 2.37 (weak conormal derivative)**.** We define

$$H^1_L(\Omega) := \{v \in H^1(\Omega) : \operatorname{div}(A\,\nabla u) \in L^2(\Omega)\},$$

and the *weak conormal derivative* $\gamma_1 u := g$ for $u \in H^1_L(\Omega)$ and g from Theorem 2.36, we obtain the bounded linear operator

$$\gamma_1 : H^1_L(\Omega) \to H^{-1/2}(\Gamma).$$

This definition is consistent with (2.8).

We obtain Green's first and second identity "for free": for $u,\, v \in H^1(\Omega)$:

$$\begin{aligned} a(u,\, v) &= (L\,u,\, v)_{L^2(\Omega)} + \langle \gamma_1 u,\, \gamma_0 u\rangle \quad \text{if } L\,u \in L^2(\Omega), \\ (L\,u,\, v)_{L^2(\Omega)} - (u,\, L\,v)_{L^2(\Omega)} &= \langle \gamma_1 v,\, \gamma_0 u\rangle - \langle \gamma_1 u,\, \gamma_0 u\rangle \quad \text{if } L\,u,\, L\,v \in L^2(\Omega). \end{aligned}$$

(If L includes $b \cdot \nabla$, then the second formula has to adapted.)

In rare cases, we need a further generalization of the conormal derivative:

Theorem 2.38. *For $u \in H^1(\Omega)$ and $f \in H^1(\Omega)^*$ with*

$$L\,u \;=\; f \quad \text{in } \mathcal{D}^*(\Omega),$$

there exists a linear functional $g \in H^{-1/2}(\Gamma)$ such that

$$a(u,\, v) \;=\; \langle f,\, v\rangle + \langle g,\, \gamma_0 v\rangle \qquad \forall v \in H^1(\Omega).$$

Furthermore, g is uniquely determined by u and f and

$$\|g\|_{H^{-1/2}(\Gamma)} \;\le\; C\big(\|u\|_{H^1(\Omega)} + \|f\|_{H^1(\Omega)^*}\big).$$

Exercise 6. Prove Theorem 2.38 (analogously to the proof of Thm. 2.36).

Definition 2.39. For $u \in H^1(\Omega)$ and $f \in H^1(\Omega)^*$ with $L\,u = f$ in $\mathcal{D}^*(\Omega)$, the *weak conormal derivative* $\gamma_1(u,\, f) \in H^{-1/2}(\Gamma)$ is given by g as from Theorem 2.38.

Warning: for a *general* function $u \in H^1(\Omega)$, the conormal derivative cannot be defined anymore. The minimal *additional* assumption for $u \in H^1(\Omega)$ seems to be $L\,u \in H^1(\Omega)^*$.

Restriction to $\Gamma_1 \subset \Gamma$. Any functional $g \in H^{-1/2}(\Gamma)$ can be restricted to a functional $g_{|\Gamma_1} \in H^{-1/2}(\Gamma_1) = \widetilde{H}^{1/2}(\Gamma_1)^*$, given by

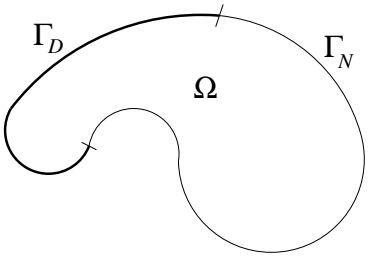
$$\langle g_{|\Gamma_1},\, \phi\rangle \;:=\; \langle g,\, \widetilde{\phi}\rangle \quad \text{for } \phi \in \widetilde{H}^{1/2}(\Gamma_1),$$

where $\widetilde{\phi} \in H^{1/2}(\Gamma)$ denotes the extension of ϕ by zero from Γ_1 to Γ (see Def. 2.22). Hence $(\gamma_1 u)_{|\Gamma_1} \in H^{-1/2}(\Gamma_1)$ is well-defined too.

2.2.3 Boundary Value Problems

Suppose that the boundary Γ of Ω splits into two disjoint parts,

$$\Gamma \;=\; \Gamma_D \cup \Gamma_N\,, \qquad \Gamma_D \cap \Gamma_N = \emptyset,$$



such that Γ_N is "Lipschitz relatively to Γ" (the special cases $\Gamma_D = \emptyset$ and $\Gamma_N = \emptyset$ are allowed). Using the notations of Sect. 2.2.1 and Sect. 2.2.2, we can define boundary value problems (BVPs).

Mixed boundary value problem Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} L\,u &= f && \text{in } \mathcal{D}^*(\Omega),\\ \gamma_0 u &= g_D && \text{in } H^{1/2}(\Gamma_D),\\ \gamma_1 u &= g_N && \text{in } H^{-1/2}(\Gamma_N), \end{aligned} \tag{2.9}$$

for given data $f \in H^1(\Omega)^*$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in H^{-1/2}(\Gamma_N)$. If $\Gamma_D = \Gamma$, (2.9) is called *first* or *Dirichlet BVP*, if $\Gamma_N = \Gamma$, *second* or *Neumann BVP*.

Third/Robin boundary value problem Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} L\,u &= f && \text{in } \mathcal{D}^*(\Omega),\\ \gamma_1 u + \beta\,\gamma_0 u &= g && \text{in } H^{-1/2}(\Gamma), \end{aligned} \tag{2.10}$$

for given data $f \in H^1(\Omega)^*$, $g \in H^{-1/2}(\Gamma)$, and $\beta \in L^\infty(\Gamma)$.

Note that $H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma)$. For $\beta = 0$, (2.10) reduces to the Neumann BVP.

Exercise 7. (a) Show that (2.9) is equivalent to

$$\text{find } u \in H^1(\Omega),\ u_{|\Gamma_D} = g_D: \quad a(u,\,v) = \langle \tilde{f},\,v\rangle \qquad \forall v \in H^1_D(\Omega),$$

where $H^1_D(\Omega) := \{v \in H^1(\Omega) : v_{|\Gamma_D} = 0\}$. How is $\tilde{f}$ defined?
(b) Derive the variational formulation which is equivalent to (2.10).

Solvability using the Lax-Milgram theorem (discussed in the Lectures *Numerical Methods for Partial Differential Equations* and *Numerical Methods for Elliptic PDEs.*) If the bilinear form $a(\cdot,\cdot)$ satisfies

$$\exists \underline{a} > 0: \quad a(v,\,v) \;\geq\; \underline{a}\,\|v\|^2_{H^1(\Omega)} \qquad \forall v \in H^1_D(\Omega), \tag{2.11}$$

then problem (2.9) has a unique solution.

A few special cases:

1. $c > 0$: condition (2.11) is fulfilled.
2. $c = 0$, $|\Gamma_D| > 0$: (2.11) fulfilled (Friedrichs' inequality).
3. $c = 0$, $\Gamma_D = \emptyset$ (pure Neumann problem): (2.11) fulfilled for $v \in H^1(\Omega)$, $\int_\Omega v\,dx = 0$ (Poincaré's inequality). Problem (2.9) is then solvable if and only if
$$\langle f, 1\rangle + \langle g_N, 1\rangle \;=\; 0.$$
In that case, the solution is unique up to an additive constant.
4. If β is uniformly positive, i.e., $\beta(x) \geq \beta_0 = \text{const} > 0$ for all $x \in \Gamma$ a.e., then the Robin problem (2.10) is uniquely solvable.

2.2.4 Fredholm Theory*

This subsection is only a minimal collection. For more details see the lecture *Integral Equations* or e.g., [McLean, Ch. 2]. Let X, Y be Banach spaces.

Definition 2.40. A bounded linear operator $B : X \to Y$ is *Fredholm* if
1. range(B) is closed in Y,
2. $\ker(B)$ and the factor space $Y_{/\text{range}(B)}$ are finite-dimensional.
The *index* of B is then defined by
$$\text{index}(B) \;:=\; \dim(\ker(B)) - \dim(Y_{/\text{range}(B)}).$$

Theorem 2.41 (Fredholm's alternative)**.** *Suppose that $B : X \to Y$ is Fredholm with* $\text{index}(B) = 0$. *Then there are two, mutually exclusive possibilities:*

(i) The homogeneous equation $B\,u = 0$ has only the trivial solution $u = 0$. In this case,
   (a) for each $f \in Y$, the inhomogeneous equation $B\,u = f$ has a unique solution $u \in X$,
   (b) for each $g \in X^*$, the adjoint[3] equation $B^*v = g$ has a unique solution $v \in Y^*$.

(ii) The homogeneous equation $B\,u = 0$ has exactly p linearly indepent solutions $u_1, \ldots, u_p$ for some finite $p \geq 1$. In this case,
   (a) the homogeneous adjoint equation $B^*v = 0$ has exactly p linearly independent solutions $v_1, \ldots, v_p$,

[3]The adjoint operator $B^* : Y^* \to X^*$ is defined by $\langle B^*f, v\rangle = \langle f, B\,v\rangle$ for $v \in X$, $f \in Y^*$ (in the complex case, one has to add conjugation).

(b) *the inhomogeneous equation* $B\,u = f$ *is solvable if and only if the right-hand side* f *satisfies* $\langle f,\, v_j\rangle = 0$ *for* $j = 1, \dots, p$,

(c) *the inhomogeneous adjoint equation* $B^* v = g$ *is solvable if and only if the right-hand side* g *satisfies* $\langle g,\, u_j\rangle = 0$ *for* $j = 1, \dots, p$.

Let $V \subset H$ be two Hilbert spaces such that V is *dense* in H and

$$\|u\|_H \le C\,\|u\|_V \qquad \forall u \in V.$$

The prominent example is $H = L^2(\Omega)$ and V a closed subspace of $H^1(\Omega)$. We *identify* H with its dual H^*. Then we may write

$$V \subset H \subset V^*,$$

and we say that H acts as a *pivot space* for V. Note that the duality product $\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{R}$ is a consistent extension of the inner product $(\cdot,\cdot)_H$:

$$\langle u,\, v\rangle \;=\; (u,\, v)_H \qquad \forall u \in V \subset V^*,\ v \in V.$$

Definition 2.42. 1. We say that a bilinear form $b(\cdot,\cdot) : V \times V \to \mathbb{R}$ fulfills a *Gårding inequality* (on V with respect to H), if there exist constants $c > 0$ and $C < \infty$ such that

$$b(v,\, v) \;\ge\; c\,\|v\|_V^2 - C\,\|v\|_H^2 \qquad \forall v \in V.$$

2. A bounded linear operator $B : V \to V^*$ is called *coercive*[4] (on V with respect to the pivot space H) if its associated bilinear form $\langle B\cdot,\cdot\rangle$ fulfills a Gårding inequality.

Theorem 2.43 (Gårding implies Fredholm)**.** *If* H *acts as a pivot space for* V*, if the bounded linear operator* $B : V \to V^*$ *is coercive on* V*, and if the inclusion map* $V \to H$ *is compact, then* B *is Fredholm with index zero.*

Example 2.44. Problem (7) is equivalent to

$$\text{find } u_0 \in H^1_D(\Omega) : \quad a(u_0,\, v) \;=\; \underbrace{\langle \tilde f,\, v\rangle - a(\mathcal{E} g_D,\, v)}_{=:\langle \bar f,\, v\rangle} \qquad \forall v \in H^1_D(\Omega),$$

where $\mathcal{E} g_D$ is a suitable extension of g_D from Γ_D to Ω; see also Exercise 7. It is further equivalent to the operator equation $B\,u_0 = \bar f$, where $B : H^1_D(\Omega) \to H^1_D(\Omega)^*$ is defined by $\langle B\,v,\, w\rangle = a(v,\, w)$. Since $a(v,\, v) + (1 + |c|)\,\|v\|^2_{L^2(\Omega)} \ge \min(\lambda_{\min}(A),\, 1)\,\|v\|^2_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$, it follows that B is coercive. With the fact that $H^1_D(\Omega)$ is compactly included in $L^2(\Omega)$, it follows that B is Fredholm with index zero, and so Fredholm's alternative holds.

[4]Note that a bilinear form $b(\cdot,\cdot)$ is called coercive (or elliptic, or positive and bounded from below), if $b(v,\, v) \ge \underline{b}\,\|v\|_V^2$. In that case, the associated operator B is often called *elliptic*.

The next theorem and corollary show that for the whole range of $c \in \mathbb{R}$, Case (ii) in Fredholm's alternative only occurs a *countable* number of times.

Theorem 2.45 (spectral theorem)**.** *Let H act as a pivot space for V, assume that H is infinite-dimensional, and that the inclusion map $V \to H$ is compact. If the bounded linear operator $S : V \to V^*$ is self-adjoint and coercive, then there exist sequences $(v_j)_{j\in\mathbb{N}} \in V$ and $(\lambda_j)_{j\in\mathbb{N}} \in \mathbb{R}$ such that*

(i) $S\,v_j = \lambda_j v_j$,

(ii) the eigenvectors $(v_j)_{j\in\mathbb{N}}$ form a complete orthonormal system in H,

(iii) the eigenvalues satisfy $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ and $\lambda_j \to \infty$ as $j \to \infty$,

(iv) $S\,u = \sum_{j=1}^{\infty} \lambda_j (u,\, v_j) v_j$ *for each $u \in V$.*

Corollary 2.46. *Let the assumptions of Theorem 2.45 hold and let $\lambda \in \mathbb{R}$. If $\lambda \notin \{\lambda_1,\, \lambda_2,\, \lambda_3, \ldots\}$, then the operator $S - \lambda I : V \to V^*$ has a bounded inverse. In particular, the equation*

$$S\,u - \lambda\,u \;=\; f$$

is uniquely solvable for all $f \in V^*$.

With this result, we can discuss cases of (2.9) where $c < 0$.

Exercise 8. Show that there exist eigenvalues $\lambda_1 \le \lambda_2 \le \cdot$ and corresponding (non-trivial) eigenfunctions $(v_j)_{j\in\mathbb{N}} \in H^1(\Omega)$ such that

$$\begin{aligned} -\mathrm{div}(A\,\nabla v_j) \;&=\; \lambda_j\, v_j \qquad && \text{in } \mathcal{D}^*(\Omega), \\ \gamma_0 v_j \;&= 0 && \text{in } H^{1/2}(\Gamma_D), \\ \gamma_1 v_j \;&= 0 && \text{in } H^{-1/2}(\Gamma_N). \end{aligned}$$

Hint: use Example 2.44 and Corollary 2.46.

In the situation of Exercise 8, one can even shown that $\lambda_1 \ge 0$. If $\Gamma_D = \Gamma$, we call λ_j a *Dirichlet eigenvalue*, if $\Gamma_N = \Gamma$ a *Neumann eigenvalue*.

We conclude: provided that $-c \notin \{\lambda_1,\, \lambda_2, \ldots\}$, problem (2.9), i.e.

$$\begin{aligned} -\mathrm{div}(A\,\nabla u) + c\,u \;&=\; f \qquad && \text{in } \mathcal{D}^*(\Omega), \\ \gamma_0 u \;&=\; g_D \qquad && \text{in } H^{1/2}(\Gamma_D), \\ \gamma_1 u \;&=\; g_N \qquad && \text{in } H^{-1/2}(\Gamma_N), \end{aligned}$$

is uniquely solvable. Otherwise, a solution exists if and only if the data satisfy

$$\langle f,\, v_j\rangle + \langle g_N,\, \gamma_0 v_j\rangle_{\Gamma_N} = \langle \gamma_1 v_j,\, g_D\rangle_{\Gamma_D} \qquad \text{for all } j \text{ with } \lambda_j = -c.$$

In particular, the Dirichlet/Neumann problem for the *Helmholtz equation* $-\Delta u - \kappa^2 u = f$ is uniquely solvable if κ^2 is not an eigenvalue. Otherwise it is only solvable under the above compatibility condition.

Chapter 3

Boundary Integral Equations

In this chapter, we develop boundary integral equations that reformulate the interior boundary value problems from Section 2.2.

To this end, we derive a *representation formula* for H^1 functions that satisfy a distributional PDE, where we make use of the fundamental solution. Since the fundamental solution acts on the whole of $\mathbb{R}^d$, we use a trick: we let the function of interest satisfy the distributional PDE *inside and outside* of the (bounded) domain Ω and allow it to be discontinuous across the interface Γ (while still being a distribution). For this setting, we derive the co-called *transmission property*. Furthermore, we obtain the representation formula, which is composed of *volume* and *and surface potentials* involving the fundamental solution.

Taking traces of the representation formula leads to the boundary integral equations, which relate the Cauchy data, i.e., the trace and the conormal derivative. We show that the involved operators are indeed integral operators and study some of their properties. In that part, we restrict ourselves mostly to the Laplace equation.

3.1 The Transmission Property

Let $\Omega \subset \mathbb{R}^d$ be a *bounded* (weakly) Lipschitz domain and define

$$\Omega^{\text{int}} := \Omega, \qquad \Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}.$$

In order to emphasize expressions related to Ω^{int}, we write γ_0^{int}, γ_1^{int} for the two trace operators from Theorem 2.25 and Sect. 2.2.2. Applying the trace theorem to Ω^{ext}, we obtain the (bounded) trace operator

$$\gamma_0^{\text{ext}} : H^1(\Omega^{\text{ext}}) \to H^{1/2}(\Gamma).$$

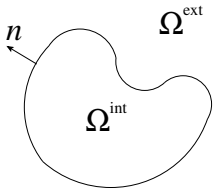


Figure 3.1: Interior/exterior domains Ω^{int}, Ω^{ext}; unit normal vector n.

The results of Sect. 2.2.2 (see Def. 2.37) hold for Ω^{ext} as well, leading to the (weak) *exterior conormal derivative*

$$\gamma_1^{\text{ext}} : H^1_L(\Omega^{\text{ext}}) \to H^{-1/2}(\Gamma).$$

However, we use a switch in the sign, such that

$$\int_{\Omega^{\text{ext}}} A\,\nabla u \cdot \nabla v\,dx = -\int_{\Omega^{\text{ext}}} \operatorname{div}(A\,\nabla u)\,v\,dx - \langle \gamma_1^{\text{ext}} u,\, \gamma_0^{\text{ext}} v\rangle \quad \forall v \in H^1(\Omega^{\text{ext}}), \tag{3.1}$$

for $u \in H^1_L(\Omega^{\text{ext}})$. If $u \in H^2(\Omega^{\text{ext}})$, then $\gamma_1^{\text{ext}} u = \gamma_0^{\text{int}}(A\,\nabla u)\cdot n$, where n is the unit normal vector outward to Ω^{int}, i.e., *inward* to Ω^{ext}.

Remark* 3.1. In the analogous way to Thm. 2.38, Def. 2.39, one obtains the generalized conormal derivative $\gamma_1^{\text{ext}}(u,\, f^{\text{ext}}) \in H^{-1/2}(\Gamma)$ for $u \in H^1(\Omega^{\text{ext}})$, $f^{\text{ext}} \in H^1(\Omega^{\text{ext}})^*$ with $L\,u = f^{\text{ext}}$ in $\mathcal{D}^*(\Omega^{\text{ext}})$.

To summarize, for a function

$$u \in H^1(\Omega^{\text{int}} \cup \Omega^{\text{ext}}) := \{u \in L^2(\mathbb{R}^d) : u_{|\Omega^{\text{int}}} \in H^1(\Omega^{\text{int}}),\ u_{|\Omega^{\text{ext}}} \in H^1(\Omega^{\text{ext}})\},$$

that fulfills

$$\begin{aligned} L\,u &= f^{\text{int}} \qquad \text{in } \mathcal{D}^*(\Omega^{\text{int}}),\\ L\,u &= f^{\text{ext}} \qquad \text{in } \mathcal{D}^*(\Omega^{\text{ext}}), \end{aligned} \tag{3.2}$$

for given functions $f^{\text{int}} \in L^2(\Omega^{\text{int}})$, $f^{\text{ext}} \in L^2(\Omega^{\text{ext}})$, we have

$$\begin{aligned} \underbrace{\int_{\Omega^{\text{int}}} A\,\nabla u \cdot \nabla v + c\,u\,v\,dx}_{=:\, a^{\text{int}}(u,\,v)} &= \int_{\Omega^{\text{int}}} f^{\text{int}}\,v\,dx + \langle \gamma_1^{\text{int}} u,\, \gamma_0^{\text{int}} v\rangle \quad \forall v \in H^1(\Omega^{\text{int}}),\\ \underbrace{\int_{\Omega^{\text{ext}}} A\,\nabla u \cdot \nabla v + c\,u\,v\,dx}_{=:\, a^{\text{ext}}(u,\,v)} &= \int_{\Omega^{\text{ext}}} f^{\text{ext}}\,v\,dx - \langle \gamma_1^{\text{ext}} u,\, \gamma_0^{\text{ext}} v\rangle \quad \forall v \in H^1(\Omega^{\text{ext}}). \end{aligned} \tag{3.3}$$

Definition 3.2. For $u \in H^1(\Omega^{\rm int} \cup \Omega^{\rm ext})$ fulfilling (3.2), we define the (composed) distribution

$$f \in H^1(\mathbb{R}^d)^* : \langle f,\, v\rangle := \int_{\Omega^{\rm int}} f^{\rm int}\, v\, dx + \int_{\Omega^{\rm ext}} f^{\rm ext}\, v\, dx \qquad \text{for } v \in H^1(\mathbb{R}^d),$$

and the *jumps*

$$[\![\gamma_0 u]\!] := \gamma_0^{\rm ext} u - \gamma_0^{\rm int} u \quad \in H^{1/2}(\Gamma), \tag{3.4}$$
$$[\![\gamma_1 u]\!] := \gamma_1^{\rm ext} u - \gamma_1^{\rm int} u \quad \in H^{-1/2}(\Gamma). \tag{3.5}$$

If $A = I$ and u piecewise smooth, then $[\![\gamma_1 u]\!] = \frac{\partial}{\partial n} u_{|\Omega^{\rm ext}} - \frac{\partial}{\partial n} u_{|\Omega^{\rm int}}$. Note that

$$\text{for } u \in H^1(\mathbb{R}^d): \quad \gamma_0^{\rm int} u = \gamma_0^{\rm ext} u, \quad [\![\gamma_0 u]\!] = 0,$$
$$\text{for } \phi \in \mathcal{D}(\mathbb{R}^d): \quad \gamma_1^{\rm int} \phi = \gamma_1^{\rm ext} \phi, \quad [\![\gamma_1 \phi]\!] = 0.$$

In such a situation, we simply write $\gamma_0 u$, $\gamma_1 \phi$ for the respective traces.

Exercise 9. For $u \in H^1(\Omega^{\rm int} \cup \Omega^{\rm ext})$, show that $u \in H^1(\mathbb{R}^d)$ if and only if $[\![\gamma_0 u]\!] = 0$.

Remark* 3.3. The identities (3.3) and Definition 3.2 of $f \in H^1(\mathbb{R}^d)^*$ can be generalized straightforwardly to $f^{\rm int} \in H^1(\Omega^{\rm int})^*$ and $f^{\rm ext} \in H^1(\Omega^{\rm ext})^*$, but one must replace the volume integrals by $\langle f^{\rm int},\, v\rangle_{\Omega^{\rm int}}$ and $\langle f^{\rm ext},\, v\rangle_{\Omega^{\rm ext}}$, and use the generalized conormal derivatives $\gamma_1^{\rm int}(u,\, f^{\rm int})$ and $\gamma_1^{\rm ext}(u,\, f^{\rm ext})$.

From (3.3) and Definition 3.2, we conclude that

$$a^{\rm int}(u,\, v) + a^{\rm ext}(u,\, v) \;=\; \langle f,\, v\rangle - \langle [\![\gamma_1 u]\!],\, \gamma_0 v\rangle \qquad \forall v \in H^1(\mathbb{R}^d). \tag{3.6}$$

Apparently, $u \in H^1(\Omega^{\rm int} \cup \Omega^{\rm ext}) \subset \mathcal{S}^*(\mathbb{R}^d) \subset \mathcal{D}^*(\mathbb{R}^d)$, so u is a distribution. Therefore, the distributional derivative $L\,u$ is in $\mathcal{S}^*(\mathbb{R}^d) \subset \mathcal{D}^*(\mathbb{R}^d)$ as well, and fulfills[1]

$$\langle L\,u,\, \varphi\rangle \;=\; \langle u,\, L\,\phi\rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d) \text{ or } \mathcal{S}(\mathbb{R}^d).$$

Definition 3.4. We define the adjoint trace operators

$$\gamma_0^* : H^{-1/2}(\Gamma) \to \mathcal{E}^*(\mathbb{R}^d) : \langle \gamma_0^* w,\, \varphi\rangle := \langle w,\, \gamma_0 \varphi\rangle \quad \text{for } \varphi \in \mathcal{E}(\mathbb{R}^d), w \in H^{-\frac{1}{2}}(\Gamma),$$
$$\gamma_1^* : H^{1/2}(\Gamma) \to \mathcal{E}^*(\mathbb{R}^d) : \quad \langle \gamma_1^* v,\, \varphi\rangle := \langle \gamma_1 \varphi,\, v\rangle \quad \text{for } \varphi \in \mathcal{E}(\mathbb{R}^d), v \in H^{1/2}(\Gamma).$$

[1] In general, L on the right-hand side has to be replaced by its adjoint L^*.

Theorem 3.5. *Suppose that* $u \in H^1(\Omega^{\mathrm{int}} \cup \Omega^{\mathrm{ext}})$ *fulfills* (3.2). *Then*

$$L\,u \;=\; f + \gamma_1^*[\![\gamma_0 u]\!] - \gamma_0^*[\![\gamma_1 u]\!] \qquad \text{in } \mathcal{D}^*(\mathbb{R}^d).$$

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be arbitrary but fixed. Then the properties of the distributional derivative imply

$$\langle L\,u,\, \varphi\rangle \;=\; \langle u,\, L\,\varphi\rangle \;=\; \int_{\Omega^{\mathrm{int}}} u\,(L\,\varphi)\,dx + \int_{\Omega^{\mathrm{ext}}} u\,(L\,\varphi)\,dx \;=:\; (*). \tag{3.7}$$

By Green's first identity (Lemma 2.34 applied in Ω^{int} and Ω^{ext} separately with $u \mapsto \varphi \in \mathcal{D} \subset H^2$ and $v \mapsto u \in H^1$), we obtain

$$(*) \;=\; a^{\mathrm{int}}(u,\, \varphi) - \langle \gamma_1\varphi,\, \gamma_0^{\mathrm{int}} u\rangle \;+\; a^{\mathrm{ext}}(u,\, \varphi) + \langle \gamma_1\varphi,\, \gamma_0^{\mathrm{ext}} u\rangle.$$

We substitute (3.6) into that identity and use Definition 3.2 of the jump:

$$(*) \;=\; \langle f,\, \varphi\rangle - \langle [\![\gamma_1 u]\!],\, \gamma_0\varphi\rangle \underbrace{-\langle \gamma_1\varphi,\, \gamma_0^{\mathrm{int}} u\rangle + \langle \gamma_1\varphi,\, \gamma_0^{\mathrm{ext}} u\rangle}_{= \langle \gamma_1\varphi,\, [\![\gamma_0 u]\!]\rangle}. \tag{3.8}$$

Combining (3.7), (3.8) as well as Definition 3.4 of the adjoint trace operators, we obtain

$$\langle L\,u,\, \varphi\rangle \;=\; \langle f,\, v\rangle - \langle \gamma_0^*[\![\gamma_1 u]\!],\, \varphi\rangle + \langle \gamma_1^*[\![\gamma_0 u]\!],\, \varphi\rangle,$$

which concludes the proof. □

Remark 3.6. Theorem 3.5 shows that $L\,u = f$ in $\mathcal{D}^*(\mathbb{R}^d)$ if and only if

$$[\![\gamma_0 u]\!] = 0 \quad \text{and} \quad [\![\gamma_1 u]\!] = 0.$$

Outlook: Our next goal is to apply a fundamental solution $\mathcal{G}$ to the identity $L\,u = f + \gamma_1^*[\![\gamma_0 u]\!] - \gamma_0^*[\![\gamma_1 u]\!]$ of Theorem 3.5, such that we get

$$u \;=\; \mathcal{G}f + \mathcal{G}\gamma_1^*[\![\gamma_0 u]\!] - \mathcal{G}\gamma_0^*[\![\gamma_1 u]\!],$$

i.e., we can *represent* u in terms of f and the Cauchy data. If we set $u_{|\Omega^{\mathrm{ext}}} = 0$ and $f^{\mathrm{ext}} = 0$, we obtain

$$u \;=\; \mathcal{G}f^{\mathrm{int}} - \mathcal{G}\gamma_1^*\gamma_0^{\mathrm{int}} u + \mathcal{G}\gamma_0^*\gamma_1^{\mathrm{int}} u.$$

This is known as Green's third identity (cf. (1.4) in the introduction) and will yield the desired boundary integral equations when applying trace operators to it.

3.2 Fundamental Solutions

Definition 3.7. A linear operator $\mathcal{G} : \mathcal{E}^*(\mathbb{R}^d) \to \mathcal{D}^*(\mathbb{R}^d)$ is called *fundamental solution* of the differential operator L iff

$$L\,\mathcal{G}u \;=\; u \;=\; \mathcal{G}Lu \qquad \forall u \in \mathcal{E}^*(\mathbb{R}^d).$$

If $\mathcal{G}$ is an integral operator with kernel $G(x,\, y) = G(x - y)$, i.e.,

$$(\mathcal{G}w)(x) \;=\; \int_{\mathbb{R}^d} G(x-y)\, w(y)\, dy \qquad \text{(for } w \text{ sufficiently smooth)}, \tag{3.9}$$

then G is called fundamental solution as well. In such a situation, $\mathcal{G}$ above is often called *volume potential* or *Newton potential*.

In the following, we provide fundamental solutions for a few specific differential operators.

- Laplace operator: $L\,u = -\Delta u$:

$$G(z) \;=\; \begin{cases} -\frac{1}{2\pi} \log|z| & \text{for } d = 2, \\ \frac{1}{4\pi}\frac{1}{|z|} & \text{for } d = 3. \end{cases}$$

- $L\,u = -\mathrm{div}(A\,\nabla u)$:

$$G(z) \;=\; \frac{1}{4\pi \det(A)} \frac{1}{\sqrt{z^\top A\, z}} \quad \text{for } d = 3.$$

- Helmholtz operator: $L\,u = -\Delta u - \kappa^2 u$:

$$G(z) \;=\; \frac{1}{4\pi} \frac{e^{-i\kappa|z|}}{|z|} \quad \text{for } d = 3,\ \kappa > 0.$$

For $d = 2$, in general, one has to use Hankel or Bessel functions. For derivations, we point to the lecture *Partial Differential Equations* as well as [Evans, Steinbach, McLean].

In all above cases, G can be shown to be a distribution fulfilling

$$\mathcal{G}w \;=\; G * w \qquad \text{for } w \in \mathcal{S}^*(\mathbb{R}^d), \tag{3.10}$$

where $*$ is the distributional convolution operator (without going into the details of its definition). Moreover,

$$\mathcal{G} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), \qquad \mathcal{G} : \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^d), \tag{3.11}$$

$$L\mathcal{G} \;=\; \mathrm{id} \;=\; \mathcal{G}L \qquad \text{in } \mathcal{S}^*(\mathbb{R}^d), \tag{3.12}$$

and

$$\langle \mathcal{G}\,\psi,\, \varphi \rangle \;=\; \langle \psi,\, \mathcal{G}\varphi \rangle \qquad \forall \psi \in \mathcal{S}^*(\mathbb{R}^d),\, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Remark* 3.8. Fundamental solutions can e.g. be derived using the Fourier transform. If $\mathcal{G}w = G * w$ with $G \in \mathcal{S}^*(\mathbb{R}^d)$, then

$$L\,\mathcal{G} = \mathcal{G}L = \mathrm{id} \quad \Longleftrightarrow \quad L\,G = GL = \delta \quad \Longleftrightarrow \quad \widehat{L}\,\widehat{G} = 1,$$

where $\widehat{L}(\xi) := \xi^\top A\,\xi + c$ is the *symbol* of the differential operator L and $\widehat{G} := \mathcal{F}G$ the Fourier transform of G. Thus, G should be the inverse Fourier transform of $1/\widehat{L}$. However, there are some technicalities, see [McLean].

In this lecture, we do not check/prove (3.11)–(3.12). However, we show that the convolution integral in (3.9) is well-defined for $w \in \mathcal{S}(\mathbb{R}^d)$.

Definition 3.9. For a compact d-dimensional domain/manifold $\mathcal{M}$, an integral kernel $k : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is called *weakly singular* if $k(x,\, y)$ is continuous at all $x \neq y \in \mathcal{M}$ and if there exist constants $C > 0$ and $\alpha > 0$ such that

$$|k(x,\, y)| \;\leq\; C\,|x - y|^{\alpha - d} \qquad \forall x \neq y \in \mathcal{M}.$$

Lemma 3.10. *For the fundamental solutions on page 33, the integral*

$$(\mathcal{G}\varphi)(x) \;=\; \int_{\mathbb{R}^d} G(x - y)\,\varphi(y)\,dy \qquad \textit{for } \varphi \in \mathcal{S}(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

exists as an improper integral.

Proof. We fix $x \in \mathbb{R}^d$ and consider the ball $B_R(x)$ with radius $R \in (0,\, \infty)$.

1. On $B_R(x)$, the integral kernel $k(x, y) := G(x - y)$ is weakly singular.

2. Hence,

$$\Big|\int_{B_R(x)} G(x - y)\,\varphi(y)\,dy\Big| \;\leq\; \|\varphi\|_{L^\infty(B_R(x))} \int_{B_R(x)} |G(x - y)|\,dy.$$

Using the estimate from Def. 3.9 and transforming to polar/spherical coordinates (around the center x), one sees that the bound is finite. This shows that the (improper) integral is absolutely convergent.

3. On the remainder $\mathbb{R}^d \setminus \overline{B}_R(x)$, G is smooth and

$$\int_{\mathbb{R}^d \setminus \overline{B}_R(x)} G(x - y)\,\varphi(y)\,dy \;=\; \int_{\mathbb{R}^d \setminus \overline{B}_R(0)} z^{-\beta}\,G(z)\,z^\beta\,\varphi(x - z)\,dz.$$

Since φ is rapidly decreasing, $z^\beta \varphi(x - z)$ has bounded L^∞-norm. Choosing β suitably, the integral over $z^{-\beta} G(z)$ can be bounded. □

Exercise 10. Work out the details of the proof of Lemma 3.10.

The technique shown in the proof of Lemma 3.10 can be used for any integral with weakly singular kernel.

3.3 Representation Formula and Potentials

As in Section 3.1, let $u \in H^1(\Omega^{\text{int}} \cup \Omega^{\text{ext}})$ fulfill (3.2), i.e.

$$L\,u = f^{\text{int}} \quad \text{in } \mathcal{D}^*(\Omega^{\text{int}}), \qquad L\,u = f^{\text{ext}} \quad \text{in } \mathcal{D}^*(\Omega^{\text{ext}}),$$

with $f^{\text{int/ext}} \in L^2(\Omega^{\text{int/ext}})$ (or $H^1(\Omega^{\text{int/ext}})^*$).

3.3.1 Representation Formula (Green's 3rd Identity)

Theorem 3.11 (representation formula)**.** *Let $u \in H^1(\Omega^{\text{int}} \cup \Omega^{\text{ext}})$ fulfill* (3.2)*. If u has compact support in $\mathbb{R}^d$, then*

$$u \;=\; \mathcal{G}f + \mathcal{G}\gamma_1^*[\![\gamma_0 u]\!] - \mathcal{G}\gamma_0^*[\![\gamma_1 u]\!]\,. \qquad \text{in } \mathcal{S}^*(\mathbb{R}^d).$$

Proof. If u has compact support, it follows immediately that

$$u \in \mathcal{S}^*(\mathbb{R}^d), \qquad L\,u \in \mathcal{S}^*(\mathbb{R}^d).$$

Furthermore, $\gamma_1^*[\![\gamma_0 u]\!] \in \mathcal{E}^*(\mathbb{R}^d) \subset \mathcal{S}^*(\mathbb{R}^d)$ and $\gamma_0^*[\![\gamma_1 u]\!] \in \mathcal{E}^*(\mathbb{R}^d) \subset \mathcal{S}^*(\mathbb{R}^d)$. Repeating the arguments from the proof of Theorem 3.5 yields

$$L\,u \;=\; f + \gamma_1^*[\![\gamma_0 u]\!] - \gamma_0^*[\![\gamma_1 u]\!] \qquad \text{in } \mathcal{S}^*(\mathbb{R}^d).$$

The desired formula now follows simply by applying $\mathcal{G}$. □

Definition 3.12 (surface potentials)**.**

$$\begin{aligned} &\textit{single layer potential} && \widetilde{V} \;:=\; \mathcal{G}\gamma_0^* : && H^{-1/2}(\Gamma) \to \mathcal{S}^*(\mathbb{R}^d) \\ &\textit{double layer potential} && \widetilde{W} \;:=\; \mathcal{G}\gamma_1^* : && H^{1/2}(\Gamma) \to \mathcal{S}^*(\mathbb{R}^d) \end{aligned}$$

Corollary 3.13. *If $u \in H^1(\Omega^{\text{int}})$ fulfills $L\,u = f^{\text{int}}$ in $\mathcal{D}^*(\Omega^{\text{int}})$, then*

$$u \;=\; \mathcal{G}f^{\text{int}} - \widetilde{W}\gamma_0^{\text{int}}u + \widetilde{V}\gamma_1^{\text{int}}u \qquad \text{in } \Omega^{\text{int}}, \tag{3.13}$$

where $\mathcal{G}f^{\text{int}}$ denotes the application of $\mathcal{G}$ to the extension of f^{int} by zero.

The (interior) representation formula (3.13) relates the solution u to the right-hand side f^{int} and the Cauchy data $(\gamma_0^{\text{int}}u,\, \gamma_1^{\text{int}}u)$. We now would like to *decode* (3.13) for smooth arguments.

Volume potential From the proof of Lemma 3.10, one can conclude that

$$(\mathcal{G}f^{\text{int}})(x) \;=\; \int_{\Omega^{\text{int}}} G(x-y)\, f^{\text{int}}(y)\, dy \qquad \text{for } f^{\text{int}} \in L^\infty(\Omega^{\text{int}}),\, x \in \overline{\Omega}^{\text{int}}\,.$$

Single layer potential For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $w \in L^\infty(\Gamma)$,

$$\begin{aligned} \underbrace{\langle \mathcal{G}\gamma_0^* w}_{=\widetilde{V}w},\, \varphi\rangle_{\mathcal{S}^*\times\mathcal{S}} &= \langle \gamma_0^* w,\, \mathcal{G}\varphi\rangle_{\mathcal{S}^*\times\mathcal{S}} \;=\; \langle w,\, \gamma_0\mathcal{G}\varphi\rangle_\Gamma \\ &= \int_\Gamma w(x) \int_{\mathbb{R}^d} G(x-y)\,\varphi(y)\,dy\,ds_x \\ &= \int_{\mathbb{R}^d}\int_\Gamma \underbrace{G(x-y)}_{G(y-x)}\, w(x)\,ds_x\,\varphi(y)\,dy. \end{aligned} \tag{3.14}$$

Warning: exchanging the two integrals is only valid if both inner integrals exist and the integral of $G(x-y)\,w(x)\,\varphi(y)$ over $\mathbb{R}^d \times \Gamma$ exists (Fubini). As we shall see in the (proof of) next lemma, this is indeed the case.

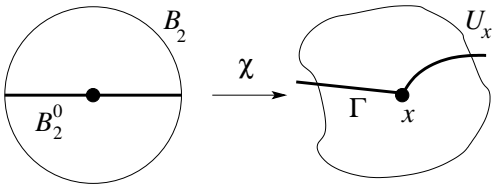
Lemma 3.14. *For the fundamental solutions on page 33,*

$$(\widetilde{V}w)(x) \;=\; \int_\Gamma G(x-y)\,w(x)\,ds_y \qquad \text{for } w \in L^\infty(\Gamma), \tag{3.15}$$

where right-hand side exists as an improper integral for all $x \in \mathbb{R}^d$. Moreover, $\widetilde{V}w \in C(\mathbb{R}^d)$ (for $w \in L^\infty(\Gamma)$).

Proof. Step 1. We show that the integral exists. Fix $w \in L^\infty(\Gamma)$ and $x \in \Gamma$ (for $x \notin \Gamma$, the integral contains no singularity and thus exists).

Recall that Ω^{int} was assumed to be weakly Lipschitz with boundary Γ. Let U_x be a neighborhood of x such that there exists a bi-Lipschitz map $\chi : \overline{B}_0 \to \overline{U}_x$ such that $\chi(\overline{B}_0^2) = \Gamma \cap \overline{U}_x$ and $\chi(0) = x$.



Due to the parametrization of curve/surface integrals,

$$\int_{\Gamma\cap U_x} G(x-y)\,w(y)\,ds_y \;=\; \int_{B_0^2} G(\chi(0)-\chi(\xi))\,\widehat{w}(\xi)\,J(\xi)\,d\xi,$$

where $\widehat{w}(\xi) = w(\chi(\xi))$ and J is the suitable Gram determinant depending on the gradient of $\chi_{|B_0^2}$. Due to the assumptions on w and χ,

$$\widehat{w} \in L^\infty(B_0^2), \qquad J \in L^\infty(B_0^2).$$

This shows that

$$\Big|\int_{\Gamma\cap U_x} G(x-y)\,w(y)\,ds_y\Big| \;\le\; \underbrace{\|J\,\widehat{w}\|_{L^\infty(B_0^2)}}_{<\infty} \underbrace{\int_{B_0^2} |G(\chi(0)-\chi(\xi))|\,d\xi}_{=:(*)}.$$

The fundamental solutions under consideration fulfill

$$|G(x-y)| \;\le\; C\,|x-y|^{\alpha-d+1} \qquad \text{for } x,\, y \in U_x\,,$$

with $C > 0$, $\alpha > 0$. Since χ and χ^{-1} are Lipschitz,

$$C^{-1}\,|0-\xi| \;\le\; |\chi(0)-\chi(\xi)| \;\le\; C\,|0-\xi| \qquad \forall \xi \in B_2^0,$$

where C is a generic constant. Using both properties, we get

$$(*) \;\le\; C \int_{B_0^2} |\chi(0)-\chi(\xi)|^{\alpha-d+1}\,d\xi \;\le\; C \int_{B_0^2} |\xi|^{\alpha-d+1}\,d\xi$$

With similar arguments than in the proof of Lemma 3.10 / Exercise 10, one sees that the $(d-1)$-dimensional integral on the right-hand side exists.

Step 2. Continuity of the integral on $\mathbb{R}^d \setminus \Gamma$ is seen rather easily. For $x \in \Gamma$ and $\widetilde{x} \in \mathbb{R}^d$, assume that U_x from Step 1 is chosen with $\operatorname{diam}(U_x) = \varepsilon$ and $\widetilde{x} \in U_x$. Splitting the joint integral in $\big|(\widetilde{V}w)(x) - (\widetilde{V}w)(\widetilde{x})\big|$ into a part (I) over $\Gamma \setminus U_x$ and a part (II) over $\Gamma \cap U_x$, using the triangle inequality and the same arguments as in Step 1 yields that part (I) $\overset{\varepsilon\to 0}{\to} 0$ and part (II) $\le C\,\varepsilon^{\alpha}$.

Step 3. Splitting the total integral over $\mathbb{R}^d \times \Gamma$ into one over $B_R \times \Gamma$ and a remainder (where $B_R \supset\supset \Omega^{\text{int}}$, using that $G(x-y)$ is weakly singular with respect to $B_R \times \Gamma$, and employing analogous techniqes as in Step 1, one can conclude that the product integral exists $\implies$ Fubini is applicable.

Step 4. (3.15) now follows from (3.14). □

Exercise 11. Complete Step 2 of the proof above.

Double layer potential Here, matters are more complicated and will become clear later on. At least, we see rather easily that for $v \in L^\infty(\Gamma)$,

$$(\widetilde{W}v)(x) \;=\; \int_\Gamma \big[A\nabla_y G(x-y)\cdot n(y)\big] v(y)\,ds_y \qquad \forall x \in \mathbb{R}^d \setminus \Gamma. \tag{3.16}$$

Summary For $u \in H^1(\Omega^{\text{int}}) \cap L^\infty(\Omega^{\text{int}})$, $f^{\text{int}} \in L^\infty(\Omega^{\text{int}})$ and $A = I$, the representation formula (3.13) decodes to

$$\begin{aligned} u(x) \;=\; \int_{\Omega^{\text{int}}} G(x-y)\,f^{\text{int}}(y)\,dy + \int_\Gamma \big[\frac{\partial}{\partial n_y} G(x-y)\big] u(y)\,ds_y \\ - \int_\Gamma G(x-y)\,\frac{\partial u}{\partial n}(x)\,ds_y \qquad \forall x \in \Omega^{\text{int}}. \end{aligned} \tag{3.17}$$

3.3.2 Further Properties of the Volume Potential

By Definition 3.7,

$$L\,\mathcal{G} f^{\text{int}} \;=\; f^{\text{int}} \qquad \text{in } \mathcal{D}^*(\Omega^{\text{int}}),$$

i.e., the *volume/Newton potential* is a solution of the inhomogeneous PDE.

Lemma 3.15. *Let $\mu_1,\ \mu_2 \in C_0^\infty(\mathbb{R}^d)$ be cutoff functions that equal 1 in a neighborhood of Ω^{int}. Then, for any $s \in \mathbb{R}$, the operator*

$$\mu_1 \mathcal{G} \mu_2 : H^{s-1}(\mathbb{R}^d) \to H^{s+1}(\mathbb{R}^d)$$

is linear and continuous. In particular, $\mu_1 \mathcal{G} \mu_2 : H^1(\mathbb{R}^d)^* \to H^1(\mathbb{R}^d)$.

Proof. Using Fourier transform, see e.g. [Steinbach, McLean]. □

Corollary 3.16. *For $f^{\text{int}} \in H^1(\Omega^{\text{int}})^*$, $\mathcal{G} f^{\text{int}} \in H^1(\Omega^{\text{int}})$, and*

$$\mathcal{G} : H^1(\Omega^{\text{int}})^* \to H^1(\Omega^{\text{int}})$$

is a linear and continuous operator.

Exercise 12. Prove Corollary 3.16 (by using the result of Lemma 3.15).

3.3.3 Properties of the Surface Potentials

From Def. 3.12 and Def. 3.4, for $\varphi \in \mathcal{E}(\mathbb{R}^d)$,

$$\langle L\,\widetilde{V} w,\, \varphi \rangle_{\mathbb{R}^d} \;=\; \langle \underbrace{L\mathcal{G}}_{=\text{id}} \gamma_0^* w,\, \varphi \rangle_{\mathbb{R}^d} \;=\; \langle w,\, \gamma_0 \varphi \rangle_\Gamma \,.$$

If we choose $\varphi \in \mathcal{D}(\Omega^{\text{int}})$, the term on the right-hand side vanishes. The analogous property holds for the double layer potential. We summarize:

Lemma 3.17. *For any $w \in H^{-1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$,*

$$L\,\widetilde{V} w \;=\; 0, \qquad L\,\widetilde{W} v \;=\; 0 \qquad \text{in } \mathcal{D}^*(\Omega^{\text{int}}) \text{ and } \mathcal{D}^*(\Omega^{\text{ext}}). \tag{3.18}$$

By their original definition, the surface potentials map into distributional spaces. Lemma 3.18 below discusses mapping properties in Sobolev spaces.

Exercise 13. Prove that $\gamma_0^* : H^{-1/2}(\Gamma) \to H^1(\mathbb{R}^d)^*$ is linear and continuous. *Hint:* It suffices (why?) to show that there exists a constant C:

$$|\langle \gamma_0^* w,\, \varphi \rangle| \;\le\; C\, \|w\|_{H^{-1/2}(\Gamma)}\, \|\varphi\|_{H^1(\mathbb{R}^d)} \qquad \forall w \in H^{-1/2}(\Gamma),\, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Lemma 3.18 (mapping properties)**.** *Let $\mu_1 \in C_0^\infty(\mathbb{R}^d)$ be a cutoff function that equals 1 in a neighborhood of $\Omega^{\rm int}$. Then*

$$\begin{aligned} &(i) \quad \mu_1 \widetilde{V} : H^{-1/2}(\Gamma) \to H^1(\mathbb{R}^d), \\ &(ii) \quad \mu_1 \widetilde{W} : H^{1/2}(\Gamma) \to H^1(\Omega^{\rm int} \cup \Omega^{\rm ext}) \end{aligned}$$

are linear and bounded operators.

Proof. (i) Let $\mu_2 \in C_0^\infty(\mathbb{R}^d)$ be a second cutoff function with $\mu_{2|\Gamma} = 1$. From the definition of γ_0^*, we see that for any $\psi \in H^{-1/2}(\Gamma)$, $\varphi \in \mathcal{E}(\mathbb{R}^d)$,

$$\langle \gamma_0^* \psi,\, \varphi \rangle \;:=\; \langle \psi,\, \gamma_0 \varphi \rangle \;=\; \langle \psi,\, \gamma_0 \mu_2 \varphi \rangle \;=\; \langle \gamma_0^* \psi,\, \mu_2 \varphi \rangle \;=\; \langle \mu_2 \gamma_0^* \psi,\, \varphi \rangle,$$

and so $\gamma_0^* \psi = \mu_2 \gamma_0^* \psi$. For $w \in H^{-1/2}(\Gamma)$,

$$\| \mu_1 \underbrace{\mathcal{G} \mu_2 \gamma_0^* w}_{= \mathcal{G}\gamma_0^* w = \widetilde{V} w} \|_{H^1(\mathbb{R}^d)} \overset{\text{Lem, 3.15}}{\leq} C \, \| \gamma_0^* w \|_{H^1(\mathbb{R}^d)^*} \overset{\text{Ex. 13}}{\leq} C \, \| w \|_{H^{-1/2}(\Gamma)} .$$

(ii) Fix $v \in H^{1/2}(\Gamma)$. Let λ be large enough ($\lambda + c > 0$), such that the Dirichlet BVP

$$\begin{aligned} Lu + \lambda u \;&=\; 0 \quad \text{in } \mathcal{D}^*(\Omega^{\rm int}), \\ \gamma_0^{\rm int} u \;&=\; v \quad \text{in } H^{1/2}(\Gamma) \end{aligned}$$

has a unique solution $u \in H^1(\Omega^{\rm int})$. As Exercise 14 below shows, there exists a (generic) constant C such that

$$\| u \|_{H^1(\Omega^{\rm int})} \;\leq\; C \, \| v \|_{H^{1/2}(\Gamma)} , \qquad \| \gamma_1^{\rm int} u \|_{H^{-1/2}(\Gamma)} \;\leq\; C \, \| v \|_{H^{1/2}(\Gamma)}. \tag{3.19}$$

We define $u_{|\Omega^{\rm ext}} := 0$ such that $u \in L^2(\mathbb{R}^d)$. Then

$$Lu \;=\; -\lambda u \quad \text{in } \mathcal{D}^*(\Omega^{\rm int}) \text{ and } \mathcal{D}^*(\Omega^{\rm ext}),$$

and so Theorem 3.11 implies

$$u \;=\; -\lambda \mathcal{G} u + \widetilde{W} \underbrace{[\![\gamma_0 u]\!]}_{= -v} - \widetilde{V} \underbrace{[\![\gamma_1 u]\!]}_{= -\gamma_1^{\rm int} u} \qquad \text{in } \mathcal{D}^*(\mathbb{R}^d).$$

Summarizing,

$$\widetilde{W} v \;=\; -u - \lambda \mathcal{G} u + \widetilde{V} \gamma_1^{\rm int} u \qquad \text{in } \mathcal{D}^*(\mathbb{R}^d). \tag{3.20}$$

To bound $\mu_1 \widetilde{W} v$, we just need to bound the H^1-norms of the (cut) individual three terms. This is done using (3.19), Lemma 3.15, and part (i); see Exercise 15 below. □

Exercise 14. Prove (3.19). *Hint:* use the estimate $\|u\|_V \leq C\,\|f\|_{V^*}$ from the Lax-Milgram theorem, where $f \in V^*$, $a(u,\,v) = \langle f,\,v\rangle \quad \forall v \in V$ with $a(\cdot,\cdot)$ coercive and bounded on the Hilbert space V.

Exercise 15. Complete the last step of the proof above, i.e., the step after (3.20). *Hint:* Estimate $\|\mu_1 \mathcal{G} u\|_{H^2(\mathbb{R}^d)}$ using Lemma 3.15 with $s = 1$ to get an intermediate bound in terms of $\|u\|_{L^2(\Omega^{\text{int}})}$.

Remark* 3.19. For $v,\, w \in L^\infty(\Gamma)$,

$$\widetilde{V} w,\ \widetilde{W} v \in C^\infty(\mathbb{R}^d \setminus \Gamma).$$

This is seen from (3.15), (3.16) and the fact that $G(x-y)$ is C^∞ for all $x \neq y \in \mathbb{R}^d$. The above property even holds for $v,\, w \in L^1(\Gamma)$.

Since the surface potentials map to H^1 and fulfill the PDE, we can apply trace operators to them.

Lemma 3.20 (traces of the surface potentials)**.** *The operators*

$$\begin{aligned} &\gamma_0^{\text{int}}\widetilde{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad &&\gamma_1^{\text{int}}\widetilde{V} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ &\gamma_0^{\text{int}}\widetilde{W} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad &&\gamma_1^{\text{int}}\widetilde{W} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \end{aligned}$$

and the corresponding ones with the exterior traces are linear and continuous.

Proof. The properties for $\gamma_0^{\text{int}}\widetilde{V}$, $\gamma_0^{\text{int}}\widetilde{W}$ follow immediately from Lemma 3.18 and the trace theorem (Thm. 2.25).
Due to (3.18) and Thm. 2.36, the interior conormal derivatives of $\widetilde{V} w$, $\widetilde{W} v$ are well-defined in $H^{-1/2}(\Gamma)$ and depend linearly and continuously on these functions. The total continuity follows now again from Lemma 3.18. □

Exercise 16. Prove the corresponding statements for the exterior operators. *Warning:* The definition of γ_1^{ext} cannot be used directly, because $\widetilde{V} w,\ \widetilde{W} v \notin H^1(\Omega^{\text{ext}})$. *Hint:* work on the *bounded* domain $\Omega^{\text{ext}} \cap B_R$ with R sufficiently large such that $\Gamma \subset \partial(\Omega^{\text{ext}} \cap B_R)$.

The next lemma shows if and how jumps of the above traces occur across the interface Γ.

Lemma 3.21 (jump relations)**.**

$$\begin{aligned} &(i) \quad [\![\gamma_0 \widetilde{V} w]\!] = 0 \qquad (ii) \quad [\![\gamma_1 \widetilde{V} w]\!] = -w \qquad \forall w \in H^{-1/2}(\Gamma), \\ &(iii) \quad [\![\gamma_0 \widetilde{W} v]\!] = v \qquad (iv) \quad [\![\gamma_1 \widetilde{W} v]\!] = 0 \qquad \forall v \in H^{1/2}(\Gamma). \end{aligned}$$

Proof. (i) follows immediately from Lemma 3.18.

(ii) For fixed $w \in H^{-1/2}(\Gamma)$, set $u := \widetilde{V}w = \mathcal{G}\gamma_0^* w$. On the one hand, Theorem 3.5 (together with (3.18)) implies that

$$Lu \;=\; 0 + \gamma_1^* \underbrace{[\![\gamma_0 u]\!]}_{=0} - \gamma_0^* [\![\gamma_1 u]\!] \qquad \text{in } \mathcal{D}^*(\mathbb{R}^d),$$

and so (using Def. 3.4),

$$\langle Lu,\, \varphi\rangle \;=\; -\langle [\![\gamma_1 u]\!],\, \gamma_0\varphi\rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

On the other hand,

$$\langle Lu,\, \varphi\rangle \;=\; \langle \underbrace{L\mathcal{G}}_{=\mathrm{id}} \gamma_0^* w,\, \varphi\rangle \;=\; \langle w,\, \gamma_0\varphi\rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Combining the two last identities yields

$$-\langle [\![\gamma_1 u]\!],\, \gamma_0\varphi\rangle \;=\; \langle w,\, \gamma_0\varphi\rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Since $\gamma_0(\mathcal{D}(\mathbb{R}^d))$ is dense in $H^{1/2}(\Gamma)$, it follows that $-[\![\gamma_1 \widetilde{V} w]\!] = w$ in $H^{-1/2}(\Gamma)$.

(iii) Let u be as in the proof of Lemma 3.18. Applying the trace operators to (3.20) yields

$$[\![\gamma_0 \widetilde{W} v]\!] \;=\; -\underbrace{[\![\gamma_0 u]\!]}_{=-v} - \lambda \underbrace{[\![\gamma_0 \mathcal{G} u]\!]}_{=0} + \underbrace{[\![\gamma_0 \widetilde{V} \gamma_1^{\mathrm{int}} u]\!]}_{=0} \;=\; v,$$

where we have used $\mu_1 \mathcal{G} u \in H^2(\mathbb{R}^d)$ and jump relation (i).

(iv) Applying the conormal derivatives to (3.20) yields

$$[\![\gamma_1 \widetilde{W} v]\!] \;=\; -\underbrace{[\![\gamma_1 u]\!]}_{=-\gamma_1^{\mathrm{int}} u} - \lambda \underbrace{[\![\gamma_1 \mathcal{G} u]\!]}_{=0} + \underbrace{[\![\gamma_1 \widetilde{V} \gamma_1^{\mathrm{int}} u]\!]}_{=-\gamma_1^{\mathrm{int}} u} \;=\; 0,$$

where we have used $\mu_1 \mathcal{G} u \in H^2(\mathbb{R}^d)$ and jump relation (ii). □

3.4 Boundary Integral Operators

Taking traces $(\gamma_0^{\mathrm{int}},\, \gamma_1^{\mathrm{int}})$ of the representation formula (3.13), we obtain two equations from the PDE $L\,u = f^{\mathrm{int}}$ in $\mathcal{D}^*(\Omega^{\mathrm{int}})$:

$$\gamma_0^{\mathrm{int}} u \;=\; \underbrace{\gamma_0^{\mathrm{int}} \mathcal{G}}_{=:\,\mathcal{N}_0} f^{\mathrm{int}} - \underbrace{\gamma_0^{\mathrm{int}} \widetilde{W}}_{=:\,-(1-\sigma)I + K} \gamma_0^{\mathrm{int}} u + \underbrace{\gamma_0^{\mathrm{int}} \widetilde{V}}_{:=\,V} \gamma_1^{\mathrm{int}} u \qquad \text{in } H^{1/2}(\Gamma),$$

$$\gamma_1^{\mathrm{int}} u \;=\; \underbrace{\gamma_1^{\mathrm{int}} \mathcal{G}}_{=:\,\mathcal{N}_1} f^{\mathrm{int}} - \underbrace{\gamma_1^{\mathrm{int}} \widetilde{W}}_{:=\,-D} \gamma_0^{\mathrm{int}} u + \underbrace{\gamma_1^{\mathrm{int}} \widetilde{V}}_{:=\,\sigma I + K'} \gamma_1^{\mathrm{int}} u \qquad \text{in } H^{-1/2}(\Gamma).$$

The function σ will be discussed in more detail in Section 3.4.3 below. For Lipschitz domains, $\sigma = 1/2$ almost everywhere on Γ; in particular, $\sigma = 1/2$ in the sense of $H^{\pm 1/2}(\Gamma)$. It will turn out that V, K, K', and D have integral representations, and are thus called *boundary integral operators*.

Definition 3.22 (boundary integral operators)**.**

name	relation	mapping property
Newton potential $\mathcal{N}_0$	$\mathcal{N}_0 := \gamma_0^{\text{int}}\mathcal{G}$	$H^1(\Omega^{\text{int}})^* \to H^{1/2}(\Gamma)$
Newton potential $\mathcal{N}_1$	$\mathcal{N}_1 := \gamma_1^{\text{int}}\mathcal{G}$	$H^1(\Omega^{\text{int}})^* \to H^{-1/2}(\Gamma)$
single layer potential V	$V := \gamma_0^{\text{int}}\widetilde{V}$	$H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$
double layer potential K	$-(1-\sigma)I + K = \gamma_0^{\text{int}}\widetilde{W}$	$H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$
adjoint double layer potential K'	$\sigma I + K' = \gamma_1^{\text{int}}\widetilde{V}$	$H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$
hypersingular integral operator D	$D := -\gamma_1^{\text{int}}\widetilde{W}$	$H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$

All these operators are linear and continuous, which follows from Lemma 3.17, Lemma 3.18, and the mapping properties of the two trace operators γ_0^{int}, γ_1^{int}.

We will study the integral representations and properties of V, K, K', D in detail. Before, however, we have a closer look to the two boundary integral equations

3.4.1 The Calderón Identities

With the notations above, the two boundary integral equations for the interior PDE read

$$\begin{bmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{bmatrix} = \underbrace{\begin{bmatrix} (1-\sigma)I - K & V \\ D & \sigma I + K' \end{bmatrix}}_{=:\mathcal{C}} \begin{bmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{bmatrix} + \begin{bmatrix} \mathcal{N}_0 f^{\text{int}} \\ \mathcal{N}_1 f^{\text{int}} \end{bmatrix}. \quad (3.21)$$

The block operator $\mathcal{C}$ is named after Calderón.

Lemma 3.23. *The (interior) Calderón operator $\mathcal{C}$ is a projection: $\mathcal{C}^2 = \mathcal{C}$.*

Proof. Let $(\varphi, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ be arbitrary but fixed. Then, due to Lemma 3.17, the function $u := \widetilde{V}\psi - \widetilde{W}\phi$ fulfills

$$Lu = 0 \quad \text{in } \mathcal{D}^*(\Omega^{\text{int}}).$$

Taking traces of u and using Def. 3.22, we get

$$\left\{ \begin{array}{rcl} \gamma_0^{\rm int} u &=& \underbrace{\gamma_0^{\rm int}\widetilde{V}}_{=V}\,\psi - \underbrace{\gamma_0^{\rm int}\widetilde{W}}_{=-(1-\sigma)I+K}\,\varphi \\ \\ \gamma_1^{\rm int} u &=& \underbrace{\gamma_1^{\rm int}\widetilde{V}}_{=\sigma I+K'}\,\psi - \underbrace{\gamma_1^{\rm int}\widetilde{W}}_{=-D}\,\varphi \end{array} \right\} \quad \Longleftrightarrow \quad \left[\begin{array}{c} \gamma_0^{\rm int} u \\ \gamma_1^{\rm int} u \end{array} \right] = \mathcal{C} \left[\begin{array}{c} \varphi \\ \psi \end{array} \right]. \tag{3.22}$$

Moreover, since u solves $Lu = 0$, the representation formula (3.13) delivers

$$\left[\begin{array}{c} \gamma_0^{\rm int} u \\ \gamma_1^{\rm int} u \end{array} \right] = \mathcal{C} \left[\begin{array}{c} \gamma_0^{\rm int} u \\ \gamma_1^{\rm int} u \end{array} \right]. \tag{3.23}$$

Inserting (3.22) into (3.23) proves the assertion. □

Exercise 17. Use the projection property of the Calderón operator $\mathcal{C}$ to show the following algebraic identities (here with $\sigma = 1/2$):

$$\begin{array}{ll} VD = (\frac{1}{2}I + K)(\frac{1}{2}I - K), & VK' = KV, \\ DV = (\frac{1}{2}I + K')(\frac{1}{2}I - K'), & K'D = DK. \end{array}$$

3.4.2 The Single Layer Potential (V)

Recall the single layer potential operator

$$V \;:=\; \gamma_0\widetilde{V} \;:\; H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma).$$

Lemma 3.24. *For $w \in L^\infty(\Gamma)$, we have the representation*

$$(Vw)(x) \;=\; \int_\Gamma G(x-y)\, w(y)\, ds_y \qquad \forall x \in \Gamma$$

as a weakly singular surface integral.

Proof. The result follows immediately from Lemma 3.14, where we showed that for $w \in L^\infty(\Gamma)$,

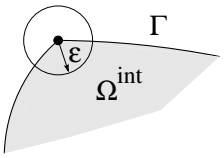
$$(\widetilde{V}w)(x) \;=\; \int_\Gamma G(x-y)\, w(y)\, ds_y \qquad \forall x \in \mathbb{R}^d,$$

and that the resulting function is continuous. □

3.4.3 The Adjoint Double Layer Potential (K')

Definition 3.25. For $x \in \Gamma$, we set

$$\begin{aligned}\sigma(x) &:= \lim_{\varepsilon\to 0} \frac{|\partial B_\varepsilon(x) \cap \Omega^{\text{int}}|}{|\partial B_\varepsilon(x)|} \\ &= \lim_{\varepsilon\to 0} \frac{1}{2(d-1)\pi\,\varepsilon^{d-1}} \int_{y\in\Omega:|x-y|=\varepsilon} ds.\end{aligned}$$



This function describes how much of the domain Ω^{int} is inside of Γ. For a (weakly) Lipschitz domain, $\sigma = 1/2$ almost everywhere on Γ (due to the fact that Γ is "differentiable" almost everywhere).

Theorem 3.26. *For $w \in L^\infty(\Gamma)$, we have the representation*

$$\gamma_1^{\text{int}} \widetilde{V} w \;=\; (\sigma I + K')w \qquad \text{in } H^{-1/2}(\Gamma),$$

with the adjoint double layer potential operator

$$(K'w)(x) \;=\; \lim_{\varepsilon\to 0} \int_{\Gamma\setminus B_\varepsilon(x)} \gamma_{1,x}^{\text{int}} G(x-y)\, w(y)\, ds_y\,.$$

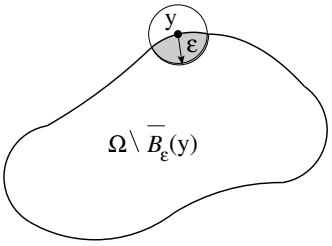
(The limit above realizes the Cauchy principal value integral.)

Proof. For the sake of a clear presentation, we present the proof only for the case $A = I$, $c = 0$. The general case can be obtained with the same techniques.

Fix $w \in L^\infty(\Gamma) \subset H^{-1/2}(\Gamma)$ and set $u := \widetilde{V}w$. Recall from Lemma 3.17 that $L\widetilde{V}w = 0$ in $\mathcal{D}^*(\Omega^{\text{int}})$. Hence, for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, due to Green's identity (Thm. 2.36),

$$\begin{aligned}\langle \gamma_1^{\text{int}} u,\, \gamma_0^{\text{int}} \varphi\rangle_\Gamma &= \int_{\Omega^{\text{int}}} \nabla u \cdot \nabla\varphi\, dx \\ &= \int_{\Omega^{\text{int}}} \nabla_x \Big(\lim_{\varepsilon\to 0} \int_{\Gamma\setminus B_\varepsilon(x)} G(x-y)\, w(y)\, ds_y \Big) \cdot \nabla\varphi(x)\, dx \\ &= \int_\Gamma w(y) \lim_{\varepsilon\to 0} \Big(\int_{\Omega^{\text{int}}\setminus B_\varepsilon(y)} \nabla_x G(x-y) \cdot \nabla\varphi(x)\, dx \Big) ds_y\,,\end{aligned}$$

where we have used that $(\widetilde{V}w)(x)$ contains a (convergent) weakly singular integral, and changed the integration order (without going into details).
For a moment, let $y \in \Gamma$ be fixed. For sufficiently small ε, the domain $\Omega^{\text{int}} \setminus \overline{B}_\varepsilon(y)$ is Lipschitz, and so Green's identity yields



$$\int_{\Omega^{\rm int}\setminus B_\varepsilon(y)} \nabla_x G(x-y)\cdot\nabla\varphi(x)\,dx$$
$$= \int_{\Gamma\setminus B_\varepsilon(y)} \Big[\frac{\partial}{\partial n_x}G(x-y)\Big]\varphi(x)\,ds_x + \int_{\partial B_\varepsilon(y)\cap\Omega^{\rm int}} \Big[\frac{\partial}{\partial n_x}G(x-y)\Big]\varphi(x)\,ds_x\,,$$

where n is outward to $\Omega\setminus\overline{B}_\varepsilon(y)$. Combining the last two identities, we get

$$\langle\gamma_1^{\rm int}u,\,\gamma_0^{\rm int}\varphi\rangle_\Gamma = \underbrace{\int_\Gamma w(y)\lim_{\varepsilon\to0}\int_{\Gamma\setminus B_{\varepsilon(y)}}\Big[\frac{\partial}{\partial n_x}G(x-y)\Big]\varphi(x)\,ds_x\,ds_y}_{=:\ (\rm I)}$$
$$+\int_\Gamma w(y)\lim_{\varepsilon\to0}\underbrace{\int_{\partial B_\varepsilon(y)\cap\Omega^{\rm int}}\Big[\frac{\partial}{\partial n_x}G(x-y)\Big]\big[\varphi(x)-\varphi(y)\big]ds_x}_{=:\ (\rm II)_\varepsilon}\,ds_y$$
$$+\int_\Gamma w(y)\lim_{\varepsilon\to0}\underbrace{\int_{\partial B_\varepsilon(y)\cap\Omega^{\rm int}}\Big[\frac{\partial}{\partial n_x}G(x-y)\Big]\varphi(y)\,ds_x}_{=:\ (\rm III)_\varepsilon}\,ds_y\,.$$

We treat the terms separately.

Term (I): Exchanging limit and integration (no details), we obtain

$$(\rm I) = \int_\Gamma \underbrace{\lim_{\varepsilon\to0}\int_{\Gamma\setminus B_\varepsilon(x)}\Big[\frac{\partial}{\partial n_x}G(x-y)\Big]w(y)\,ds_y}_{=(K'w)(x)}\ \varphi(x)\,ds_x = \langle K'w,\,\gamma_0^{\rm int}\varphi\rangle_\Gamma\,.$$

Term $(\rm II)_\varepsilon$:

$$|(\rm II)_\varepsilon| \le \underbrace{\max_{x\in\partial B_\varepsilon(y)\cap\Omega^{\rm int}}|\varphi(x)-\varphi(y)|}_{\xrightarrow{\varepsilon\to0}\ 0}\ \underbrace{\int_{\partial B_\varepsilon(y)\cap\Omega^{\rm int}}\Big|\frac{\partial}{\partial n_x}G(x-y)\Big|\,ds_x}_{=:\ (*)_\varepsilon}$$

We show now that $(*)_\varepsilon$ is bounded.
We first compute $\nabla_x G(x-y)$ (for $L=-\Delta$):

$$d=2:\quad \nabla_x G(x-y) = -\frac{1}{2\pi}\nabla_x\log|x-y| = -\frac{1}{2\pi}\frac{1}{|x-y|}\frac{x-y}{|x-y|}\,,$$
$$d=3:\quad \nabla_x G(x-y) = \frac{1}{4\pi}\nabla_x\frac{1}{|x-y|} = -\frac{1}{4\pi}\frac{1}{|x-y|^2}\frac{x-y}{|x-y|}\,.$$

Since n is outward to $\Omega \setminus \overline{B_\varepsilon(y)}$, $n_x = \frac{y-x}{|x-y|}$ for $x \in \partial B_\varepsilon(y) \cap \Omega^{\text{int}}$, and so

$$\frac{\partial}{\partial n_x} G(x,\, y) = \frac{1}{2\pi(d-1)} \frac{x-y}{|x-y|^d} \cdot \frac{x-y}{|x-y|} = \frac{1}{2\pi(d-1)} \, |x-y|^{1-d}.$$

Hence,

$$(*)_\varepsilon = \frac{1}{2\pi(d-1)} \int_{\partial B_\varepsilon(y) \cap \Omega^{\text{int}}} \underbrace{|x-y|}_{=\varepsilon}^{1-d} \, ds_x \leq \frac{1}{2\pi(d-1)} \int_{\partial B_\varepsilon(y)} \varepsilon^{1-d} \, ds_x = 1.$$

Therefore, $|(\mathbb{II})_\varepsilon| \overset{\varepsilon \to 0}{\longrightarrow} 0$.

Term $(\mathbb{III})_\varepsilon$: With the computation above, we get

$$\begin{aligned} (\mathbb{III})_\varepsilon &= \int_{\partial B_\varepsilon(y) \cap \Omega^{\text{int}}} \Big[\frac{\partial}{\partial n_x} G(x-y) \Big] \varphi(y) \, ds_x \\ &= \varphi(y) \int_{\partial B_\varepsilon(y) \cap \Omega^{\text{int}}} \frac{1}{2\pi(d-1)} \underbrace{|x-y|}_{=\varepsilon}^{1-d} \, ds_x \overset{\varepsilon \to 0}{\longrightarrow} \varphi(y)\, \sigma(y). \end{aligned}$$

Adding up the results for all the terms and using that $u = \widetilde{V}$ yields

$$\langle \gamma_1^{\text{int}} \widetilde{V} w,\, \gamma_0^{\text{int}} \varphi \rangle_\Gamma = \langle K' w,\, \gamma_0^{\text{int}} \varphi \rangle_\Gamma + 0 + \int_\Gamma w(y)\, \sigma(y)\, \varphi(y) \, ds_y \,.$$

The assertion now follows from the density of $\gamma_0^{\text{int}}(\mathcal{D}(\mathbb{R}^d))$ in $H^{1/2}(\Gamma)$. □

3.4.4 The Double Layer Potential (K)

Theorem 3.27. *For $v \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$, we have the representation*

$$(\gamma_0^{\text{int}} \widetilde{W} v)(x) = (-1 + \sigma(x)) + (Kv)(x) \qquad \text{for } x \in \Gamma,$$

with σ as in Definition 3.25 and the double layer potential operator

$$(Kv)(x) = \lim_{\varepsilon \to 0} \int_{\Gamma \setminus B_\varepsilon(x)} \big[\gamma_{1,y}^{\text{int}} G(x-y) \big] v(y) \, ds_y \,.$$

Proof. Works with similar techniques as the proof of Theorem 3.26 □

Exercise 18. Show that for $v \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$ and $w \in L^\infty(\Gamma)$,

$$\langle w,\, Kv \rangle_\Gamma = \langle K' w,\, v \rangle_\Gamma \,. \tag{3.24}$$

Hint: use Theorems 3.26 and 3.27.

From Exercise 18, one can easily conclude by a density argument that (3.24) holds also for $v \in H^{1/2}(\Gamma)$ and $w \in H^{-1/2}(\Gamma)$, which means that K' is *indeed* the adjoint operator to K.

3.4.5 The Hypersingular Integral Operator D

Recall that

$$D \;=\; -\gamma_1^{\rm int}\widetilde{W} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

Mapping $H^{1/2}$ to $H^{-1/2}$, this operator has the flavor of a differential operator (of order 1). It is hard to find an integral representation for it.

First we sketch that a straightforward technique fails. For $L = -\Delta$, for a smooth function v and $\widetilde{x} \in \Omega^{\rm int}$, one can show (using the computations from the proof of Theorem 3.26)

$$(\widetilde{W}v)(\widetilde{x}) \;=\; \frac{1}{2\pi(d-1)} \lim_{\varepsilon\to 0} \int_{\Gamma\setminus B_\varepsilon(\widetilde{x})} \frac{(\widetilde{x}-y)\cdot n_y}{|\widetilde{x}-y|^d}\, v(y)\, ds_y\,.$$

In order to get $(Dv)x$, we have to apply $-n_x \cdot \nabla_x$ to the above expression and send $\widetilde{x}$ to $x \in \Gamma$. Exchanging *formally* the two limits and computing the normal derivative yields

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi(d-1)} \int_{\Gamma\setminus B_\varepsilon(x)} \Big[-\frac{n_x\cdot n_y}{|x-y|^d} + d\frac{(x-y)\cdot n_x (x-y)\cdot n_y}{|x-y|^{d+2}}\Big] v(y)\, ds_y\,.$$

However, the above limit (realizing a possible Cauchy principal value) does *not* exist in general, because the kernel is so ("hyper") singular.

Lemma 3.28. *If $L = -\Delta$, the hypersingular integral operator admits the following representation for $v \in H^{1/2}(\Gamma) \cap C(\Gamma)$:*

$$(Dv)(x) \;=\; -\frac{\partial}{\partial n_x} \int_\Gamma \Big[\frac{\partial}{\partial n_y} G(x-y)\Big]\big[v(y)-v(x)\big] ds_y \quad \forall x \in \Gamma,$$

to be understood as a Cauchy principal value integral.

Proof. See, e.g., [Steinbach, Sect. 6.5] □

In the next chapter, we shall give an alternative representation of D which is very practical in implementations of boundary element methods.

3.4.6 Further Mapping Properties*

The operators V, K, K', and D are so-called *pseudo-differential operators* of integer order. For Lipschitz domains and for any $s \in [-1/2, 1/2]$,

$$\begin{aligned} &V : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)\,,\\ &K : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)\,, \qquad K' : H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)\,,\\ &D : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma) \end{aligned}$$

are continuous mappings. Hence, K and K' are pseudo-differential operators of order 0, V is of order $+1$ (a smoothing operator), and D is of order -1 (a differential type operator). A proof is found in [Costabel].

3.5 Boundary Integral Equations

3.5.1 The Interior Dirichlet Problem

Theorem 3.29. *Let $f^{\text{int}} \in L^2(\Omega^{\text{int}})$ (or $H^1(\Omega^{\text{int}})^*$) and $g_D \in H^{1/2}(\Gamma)$ be given.*

(i) If $u \in H^1(\Omega^{\text{int}})$ is a solution of the interior Dirichlet problem

$$\left.\begin{aligned} L\,u &= f^{\text{int}} && \text{in } \mathcal{D}^*(\Omega^{\text{int}}) \\ \gamma_0^{\text{int}} u &= g_D && \text{in } H^{1/2}(\Gamma) \end{aligned}\right\} \tag{3.25}$$

then the conormal derivative $t := \gamma_1^{\text{int}} u \in H^{-1/2}(\Gamma)$ is a solution of the boundary integral equation

$$Vt \;=\; (\sigma I + K) g_D - \mathcal{N}_0 f^{\text{int}} \qquad \text{in } H^{1/2}(\Gamma), \tag{3.26}$$

and u has the integral representation

$$u \;=\; \mathcal{G} f^{\text{int}} - \widetilde{W} g_D + \widetilde{V} t \qquad \text{in } \Omega^{\text{int}}. \tag{3.27}$$

(ii) Conversely, if $t \in H^{-1/2}(\Gamma)$ is a solution of the boundary integral equation (3.26), then formula (3.27) defines a solution $u \in H^1(\Omega^{\text{int}})$ of the interior Dirichlet problem (3.25).

Proof. (i) The fact that (3.25) implies (3.27) has been shown in Corollary 3.13. The (interior) Calderón identities (3.21) imply

$$\underbrace{\gamma_0^{\text{int}} u}_{=g_D} \;=\; ((1-\sigma)I - K) \underbrace{\gamma_0^{\text{int}} u}_{=g_D} \;+\; V \underbrace{\gamma_1 u}_{=t} \;+\; \mathcal{N}_0 f^{\text{int}},$$

from which we conclude (3.26).

(ii) Suppose that $t \in H^{-1/2}(\Gamma)$ solves (3.26) and that u is given by (3.27). The results of Section 3.3 guarantee that $u \in H^1(\Omega^{\text{int}})$ and that

$$L\mathcal{G} f^{\text{int}} = f^{\text{int}} \quad \text{in } \mathcal{D}^*(\Omega^{\text{int}}), \qquad L\widetilde{W} g_D = L\widetilde{V} t = 0 \quad \text{in } \mathcal{D}^*(\Omega^{\text{int}}).$$

Hence, the distributional PDE $Lu = f$ in $\mathcal{D}^*(\Omega^{\text{int}})$ holds. Finally,

$$\gamma_0 u \overset{(3.27)}{=} \underbrace{\gamma_0 \mathcal{G}}_{=\mathcal{N}_0} f^{\text{int}} - \underbrace{\gamma_0 \widetilde{W}}_{=(-1+\sigma)I+K} g_D \;+\; \underbrace{\gamma_0 \widetilde{V}}_{=V} t \overset{(3.26)}{=} g_D\,.$$

□

3.5.2 The Interior Steklov-Poincaré Operator

Assume that the interior Dirichlet problem (3.25) with $f = 0$ has a unique solution $u(g_D)$ for every $g_D \in H^{1/2}(\Gamma)$ (this is e.g. the case for $c \geq 0$. We denote the corresponding conormal derivative by

$$S^{\text{int}}\, g_D := \gamma_1^{\text{int}} u(g_D).$$

The corresponding operator S is called *Steklov-Poincaré* operator of the interior Dirichlet problem.

Exercise 19. Show that under the above assumption $c \geq 0$, the operator $S^{\text{int}} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is linear and continuous.

As we will show later on, the Steklov-Poincaré operator can be expressed in terms of the four boundary integral operators.

3.5.3 The Interior Neumann Problem

Theorem 3.30. *Let $f^{\text{int}} \in L^2(\Omega^{\text{int}})$ (or $H^1(\Omega^{\text{int}})^*$) and $g_N \in H^{-1/2}(\Gamma)$ be given.*

(i) If $u \in H^1(\Omega^{\text{int}})$ is a solution of the interior Neumann problem

$$\left.\begin{array}{rcll} Lu & = & f^{\text{int}} & \text{in } \mathcal{D}^*(\Omega^{\text{int}}) \\ \gamma_1^{\text{int}} u & = & g_N & \text{in } H^{-1/2}(\Gamma) \end{array}\right\} \tag{3.28}$$

then the trace $v := \gamma_0 u$ is a solution of the boundary integral equation

$$D\, v = \big[(1-\sigma)I - K'\big] g_N - \mathcal{N}_1 f^{\text{int}} \qquad \text{in } H^{-1/2}(\Gamma), \tag{3.29}$$

and u has the integral representation

$$u = \mathcal{G} f^{\text{int}} - \widetilde{W} v + \widetilde{V} g_N \qquad \text{in } \Omega^{\text{int}}. \tag{3.30}$$

(ii) Conversely, if $v \in H^{1/2}(\Gamma)$ is a solution of the boundary integral equation (3.29), then formula (3.30) defines a solution $u \in H^1(\Omega^{\text{int}})$ of the interior Neumann problem (3.28).

Exercise 20. Prove Theorem 3.30. *Hint:* analogous to the proof of Theorem 3.29.

3.5.4 Mixed Boundary Value Problems*

For simplicity, we treat here only the case $f^{\text{int}} = 0$.

For $\partial\Omega^{\text{int}} = \Gamma = \Gamma_D \cup \Gamma_N$, let $g_D \in H^{1/2}(\Gamma_D)$ and $g_N \in H^{-1/2}(\Gamma_N)$ be given. Let $u \in H^1(\Omega^{\text{int}})$ be the solution of the mixed BVP

$$\left.\begin{array}{rcll} Lu & = & 0 & \text{in } \mathcal{D}^*(\Omega^{\text{int}}) \\ \gamma_0^{\text{int}} u & = & g_D & \text{in } H^{1/2}(\Gamma_D) \\ \gamma_1^{\text{int}} u & = & g_N & \text{in } H^{-1/2}(\Gamma_N) \end{array}\right\} \tag{3.31}$$

1. Recall that functions in $H^{1/2}(\Gamma_D)$ can be extended to functions in $H^{1/2}(\Gamma)$. Therefore, there exists

$$\bar{g}_D \in H^{1/2}(\Gamma), \quad \bar{g}_{D|\Gamma_D} = g_D$$

2. Recall that $H^{-1/2}(\Gamma) = \widetilde{H}^{1/2}(\Gamma)^*$ and that any functional in $H^{-1/2}(\Gamma)$ can be *restricted* to one in $H^{-1/2}(\Gamma_N)$. Therefore, there exists a functional

$$\bar{g}_N \in H^{-1/2}(\Gamma): \quad \langle \bar{g}_N, v\rangle_\Gamma = \langle g_N, v\rangle \qquad \forall v \in \widetilde{H}^{1/2}(\Gamma_N).$$

(Note that neither the extension $\bar{g}_D$ nor $\bar{g}_N$ are unique.)

Theorem 3.31. *(i) Let $u \in H^1(\Omega^{\text{int}})$ be a solufion of (3.31) and let $\bar{g}_D$ and $\bar{g}_N$ be as above. Then the functions*

$$\begin{array}{rcll} v_0 & := & \gamma_0^{\text{int}} u - \bar{g}_D & \in \widetilde{H}^{1/2}(\Gamma_D), \\ t_0 & := & \gamma_1^{\text{int}} u - \bar{g}_N & \in \widetilde{H}^{-1/2}(\Gamma_N) = H^{1/2}(\Gamma_N)^* \end{array}$$

fulfill

$$\begin{bmatrix} D & K' \\ -K & V \end{bmatrix} \begin{bmatrix} v_0 \\ t_0 \end{bmatrix} = \begin{bmatrix} -D & (1-\sigma)I - K' \\ \sigma I + K & -V \end{bmatrix} \begin{bmatrix} \bar{g}_D \\ \bar{g}_N \end{bmatrix}, \tag{3.32}$$

as a system of boundary integral equations in $\widetilde{H}^{1/2}(\Gamma_D)^* \times \widetilde{H}^{-1/2}(\Gamma_N)^*$, and u has the integral representation

$$u = \mathcal{G}^{\text{int}} f^{\text{int}} - \widetilde{W}(\bar{g}_D + v_0) + \widetilde{V}(\bar{g}_N + t_0) \qquad \text{in } \Omega^{\text{int}}. \tag{3.33}$$

(ii) Conversely, if $(v_0, t_0) \in \widetilde{H}^{1/2}(\Gamma_D) \times \widetilde{H}^{-1/2}(\Gamma_N)$ solve (3.32), then formula (3.33) defines a solution $u \in H^1(\Omega^{\text{int}})$ of (3.31).

3.5.5 Second Kind Integral Equations*

The single layer boundary integral equation (3.26),

$$Vt \;=\; (\sigma I + K)g_D - \mathcal{N}_0 f^{\text{int}}$$

for the interior Dirichlet problem and the hypersingular boundary integral equation (3.29),

$$Dv \;=\; [(1-\sigma)I - K']g_N = \mathcal{N}_1 f^{\text{int}}$$

of the interior Neumann problem are both *1st kind integral equations*. They were both derived from using one (specific) line of the interior Calderón identity (3.21).

Using the respective other line, leads to the *2nd kind boundary integral equation*

$$[(1-\sigma)I - K']t \;=\; Dg_D + \mathcal{N}_1 f^{\text{int}} \qquad \text{in } H^{-1/2}(\Gamma)$$

for the conormal derivative $t = \gamma_1^{\text{int}} u$ of the interior Dirichlet problem with Dirichlet datum g_D and to the *2nd kind boundary integral equation*

$$(\sigma I + K)v \;=\; Vg_N + \mathcal{N}_0 f^{\text{int}} \qquad \text{in } H^{1/2}(\Gamma)$$

for the trace $v = \gamma_0^{\text{int}} u$ of the interior Neumann problem with Neumann datum g_N. Similarly to Theorems (3.29) and (3.30), the overall solution u can be reconstructed by a representation formula in both cases.

3.6 Exterior Problems

We are interested in solutions of the (for simplicity) homogeneous exterior PDE

$$Lu \;=\; 0 \qquad \text{in } \mathcal{D}^*(\Omega^{\text{ext}}). \tag{3.34}$$

Corollary 3.32. *If $u \in H^1(\Omega^{\text{ext}})$ fulfills $Lu = 0$ in $\mathcal{D}^*(\Omega^{\text{ext}})$, then*

$$u \;=\; \widetilde{W}\gamma_0^{\text{ext}} u - \widetilde{V}\gamma_1^{\text{ext}} u \qquad \text{in } \Omega^{\text{ext}}. \tag{3.35}$$

Proof. Choosing $f^{\text{int}} = f^{\text{ext}} = 0$ and $u_{|\Omega^i ntr} := 0$, we obtain from Theorem 3.11 that

$$u \;=\; \widetilde{W} \underbrace{[\![\gamma_0 u]\!]}_{=\gamma_0^{\text{ext}} u - 0} - \widetilde{V} \underbrace{[\![\gamma_1 u]\!]}_{=\gamma_1^{\text{ext}} u - 0} \qquad \text{in } \mathcal{S}^*(\mathbb{R}^d).$$

In particular,

$$(u,\ \varphi)_{L^2(\Omega^{\rm ext})} \ = \ (\widetilde{W}\gamma_0^{\rm ext}u - \widetilde{V}\gamma_1^{\rm ext}u,\ \varphi)_{L^2(\Omega^{\rm ext})} \qquad \forall \varphi \in C_0^\infty(\Omega^{\rm ext}).$$

Since $C_0^\infty(\Omega^{\rm ext})$ is dense in $L^2(\Omega^{\rm ext})$, identity (3.35) holds in the sense of $L^2(\Omega^{\rm ext})$. □

Warning: if the coefficient c in the differential operator L is zero or negative, then $H^1(\Omega^{\rm ext})$ is (usually) *not used* as solution space, because it has the "wrong" decay behavior towards infinity for physically relevant problems. Hence, for $c \geq 0$, we need a different solution space. We shall discuss in detail the case $c = 0$ and comment only briefly on the case $c < 0$. For a more comprehensive treatment of exterior problems we refer [Steinbach, Sect. 7.5], [McLean, p. 234ff], and [Sauter/Schwab] (where each reference has its own particular focus).

3.6.1 The Exterior Laplace Problem

Throughout this section, we assume that

$$L = -\Delta \qquad (A = I, c = 0).$$

Definition 3.33 (space of bounded energy)**.** For $R > 0$ define $\Omega_R^{\rm ext} := \Omega^{\rm ext} \cap B_R(0)$ and set

$$H^1_{\rm loc}(\Omega^{\rm ext}) \ := \ \big\{v \in L^1_{\rm loc}(\overline{\Omega}^{\rm ext}) : \forall R > 0 : v_{|\Omega_R^{\rm ext}} \in H^1(\Omega_R^{\rm ext})\big\}.$$

The *space of bounded energy* (for the exterior Laplace problem) is given by

$$H^1_E(\Omega^{\rm ext}) \ := \ \big\{v \in H^1_{\rm loc}(\Omega^{\rm ext}) : \int_{\Omega^{\rm ext}} |\nabla v|^2\, dx < \infty\big\}.$$

For $v,\ w \in H^1_E(\Omega^{\rm ext})$ we can still *evaluate* the exterior bilinear form

$$a^{\rm ext}(v,\ w) \ = \ \int_{\Omega^{\rm ext}} \nabla v \cdot \nabla w\, dx.$$

Lemma 3.34 (without proof)**.** *The space $H^1_E(\Omega^{\rm ext})$ equipped with norm*

$$\|v\|_{H^1_E(\Omega^{\rm ext})} := \Big(\int_{\Omega^{\rm ext}} |\nabla v|^2 + \frac{|v|^2}{1+\rho(x)^2}\, dx\Big)^{1/2}$$

is a Hilbert space, where

$$\rho(x) \ := \ \begin{cases} |x|\ \log|x| & \text{if } d = 2, \\ |x| & \text{if } d = 3. \end{cases}$$

Hence $H^1_E(\Omega^{\rm ext})$ *can be considered a* weighted *Sobolev space. Furthermore,*

$$H^1_E(\Omega^{\rm ext}) \;=\; \overline{C_0^\infty(\overline{\Omega}^{\rm ext})}^{|\cdot|_{H^1(\Omega^{\rm ext})}},$$

and for $d = 3$*, the seminorm* $|\cdot|_{H^1(\Omega^{\rm ext})}$ *is equivalent to* $\|\cdot\|_{H^1_E(\Omega^{\rm ext})}$.

Exercise 21. Assume $d = 3$ and show that if $u \in C^1(\overline{\Omega}^{\rm ext})$ fulfills

$$|u(x)| \;=\; \mathcal{O}(|x|^{-1}) \qquad \text{as } |x| \to \infty,$$

then $u \in H^1_E(\Omega^{\rm ext})$.

Now is a good time to discuss the behavior of the two surface potentials at infinity (which we didn't do so far).

Decay Behavior of the Surface Potentials

Theorem 3.35. *Let* $L = -\Delta$ *and assume that for* $d = 2$*, we use the fundamental solution kernel* $G(x) = -1/(2\pi)\log|x|$*. Then for* $v \in H^{1/2}(\Gamma)$ *and* $w \in H^{-1/2}(\Gamma)$*,*

$$(\widetilde{V}w)(x) \;=\; \left\{ \begin{array}{ll} -\frac{1}{2\pi}\,\langle w,\,1\rangle_\Gamma\,\log|x| + \mathcal{O}(|x|^{-1}) & \text{if } d=2, \\ \mathcal{O}(|x|^{-1}) & \text{if } d=3. \end{array} \right\} \quad \text{as } |x| \to \infty,$$

$$(\widetilde{W}v)(x) \;=\; \mathcal{O}(|x|^{1-d}) \quad \text{as } |x| \to \infty.$$

In general,

$$\begin{aligned} \widetilde{W}v \;&\in\; H^1_E(\Omega^{\rm ext}), \\ \widetilde{V}w \;&\in\; H^1_E(\Omega^{\rm ext}) \quad \left\{ \begin{array}{l} \text{if } d=2 \text{ and } \langle w,\,1\rangle_\Gamma = 0, \\ \text{or if } d=3. \end{array} \right. \end{aligned}$$

Proof. Assume $0 \notin \overline{\Omega}^{\rm ext}$ and $x \in \Omega^{\rm ext}$ with $|x| > \max(1, 2\mathrm{diam}(\Omega^{\rm int}))$. Then, for any $y \in \Gamma$,

$$\begin{aligned} &|x| \;\le\; |x-y| + |y| \;\le\; |x-y| + \mathrm{diam}(\Omega^{\rm int}) \;\le\; |x-y| + \tfrac{1}{2}|x|, \\ &\text{and so} \quad |x-y| \;\ge\; \tfrac{1}{2}|x|. \end{aligned}$$

(a) Single layer potential for $d = 3$:

$$|(\widetilde{V}w)(x)| \;=\; \Big|\int_\Gamma G(x-y)\,w(y)\,ds_y\Big| \;\le\; \underbrace{\|G(x-\cdot\|_{H^{1/2}(\Gamma)}}_{\le C_{\rm tr}\,\|G(x-\cdot)\|_{H^1(\Omega^{\rm int})}}\,\|w\|_{H^{-1/2}(\Gamma)}$$

A short computation reveals that

$$\|G(x-\cdot\|^2_{H^1(\Omega^{\rm int})} \;=\; c_1 \int_{\Omega^{\rm int}} \underbrace{|x-y|^{-2}}_{\leq 4\,|x|^{-2}}\, dy + c_2 \int_{\Omega^{\rm int}} \underbrace{|x-y|^{-4}}_{\leq 16\,|x|^{-4}}\, dy \;\leq\; C\,|x|^{-2}\,.$$

(b) Single layer potential for $d=2$. We choose $\bar{y}\in\Omega^{\rm int}$. A Taylor expansion gives

$$\log|x-y| \;=\; \log|x| + \frac{(\bar{y}-x)\cdot y}{|x-\bar{y}|} \qquad \text{for some } \bar{y}\in\mathbb{R}^d.$$

Therefore,

$$\begin{aligned}(\widetilde{V}w)(x) \;&=\; -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,w(y)\,ds_y \\ &=\; -\frac{1}{2\pi}\log|x| \underbrace{\int_\Gamma w(y)\,ds_y}_{=\langle w,1\rangle_\Gamma} - \frac{1}{2\pi}\underbrace{\int_\Gamma \frac{(\bar{y}-x)\cdot y}{|x-\bar{y}|}\,w(y)\,ds_y}_{=\;(*)}\end{aligned}$$

Analogously to (a), one shows that $|(*)|\leq C\,\|w\|_{H^{-1/2}(\Gamma)}|x|^{-1}$.
(c) Double layer potential for $d=2$ and $d=3$. Recall that for x as above,

$$(\widetilde{W}v)(x) \;=\; \int_\Gamma \frac{\partial}{\partial n_y} G(x-y)\,v(y)\,ds_y\,.$$

As we showed in the proof of Theorem 3.26,

$$\frac{\partial}{\partial n_y}G(x-y) = \frac{1}{2\pi(d-1)}|x-y|^{1-d}\,.$$

Hence, with the above assumptions on x and y,

$$|(\widetilde{W}v)(x)| \;\leq\; \big\|\frac{\partial}{\partial n_y}G(x-\cdot)\big\|_{L^2(\Gamma)}\,\|v\|_{L^2(\Gamma)} \;\leq\; C\,|x|^{1-d}\,\|v\|_{L^2(\Gamma)}\,.$$

(d) The $H^1_E(\Omega^{\rm ext})$-membership properties of the surface potentials now follows from Exercise 21. □

Representation Formula The trace operator $\gamma_0^{\rm ext}$ can be extended to $H^1_E(\Omega^{\rm ext})$, and the conormal derivative $\gamma_1^{\rm ext}u$ can at least be (well-)defined for $u\in H^1_E(\Omega^{\rm ext})$, $\Delta u=0$ in $\mathcal{D}^*(\Omega^{\rm ext})$, such that

$$\int_{\Omega^{\rm ext}} \nabla u\cdot\nabla v\,dx \;=\; -\langle\gamma_1^{\rm ext}u,\,\gamma_0^{\rm ext}v\rangle \qquad \forall v\in H^1_E(\Omega^{\rm ext}). \tag{3.36}$$

Also, the transmission property from Section 3.1 and the representation formulae from Section 3.3 can be extended to the larger space.

Theorem 3.36 (without proof)**.** *Let $d = 3$. If $u \in H^1_E(\Omega^{\text{ext}})$ fulfills $-\Delta u = 0$ in $\mathcal{D}^*(\Omega^{\text{ext}})$, then*

$$u \;=\; \widetilde{W}\gamma_0^{\text{ext}}u - \widetilde{V}\gamma_1^{\text{ext}}u \qquad \text{in } \Omega^{\text{ext}}.$$

For $d = 2$, a suitable constant term has to be added to the right-hand side (namely $\langle w_{\text{eq}}, \gamma_0^{\text{ext}}u\rangle_\Gamma$, cf. Def. 3.42 below).

Remark 3.37. For $d = 2$, $L = -\Delta$, and $\langle w, 1\rangle_\Gamma \neq 0$, the single layer potential $\widetilde{V}w$ *grows logarithmically* towards infinity as $|x| \to \infty$. This is a particuliarity of the two-dimensional Laplace operator, reflected by in the following facts:

- For $d = 2$, the fundamental solution kernel itself grows towards infinity.
- One can show that

  $$1 \in H^1_E(\Omega^{\text{ext}}) \quad \text{for } d = 2$$

  (but $1 \notin H^1_E(\Omega^{\text{ext}})$ if $d = 3$). Hence, for $u \in H^1_E(\Omega^{\text{ext}})$ with $-\Delta u = 0$ in $\mathcal{D}^*(\Omega^{\text{ext}})$, we obtain (from (3.36) with $v = 1$) the condition

  $$\langle \gamma_1^{\text{ext}}u, 1\rangle_\Gamma \;=\; 0. \tag{3.37}$$

  The analogous condition $\langle \gamma_1^{\text{int}}u, 1\rangle_\Gamma = 0$ holds for the interior Laplace problem (for either $d = 2$ or 3). One may say that in 3D, the space $H^1_E(\Omega^{\text{ext}})$ owns something like a Dirichlet condition at infinity, which is not present in 2D. Correspondingly, a solution of the exterior Neumann problem in 2D is only unique up to constants, whereas the exterior Neumann problem in 3D is uniquely solvable.
- Comparing the compatibility condition (3.37) with Theorem 3.35, we see that the term $\langle w, 1\rangle_\Gamma$ in front of $\log|x|$ vanishes anyway if w is the conormal derivative of a solution.

3.6.2 The Exterior Helmholtz Problem*

For $L = -\Delta - \kappa^2 I$, $d = 3$, one often uses the *Sommerfeld radiation conditions*

$$|u(x)| \;=\; \mathcal{O}(|x|^{-1}), \qquad \left|\frac{\partial u}{\partial \omega} - i\kappa u\right| \;=\; \mathcal{O}(|x|^{-2}) \qquad \text{as } |x| \to \infty,$$

where $\omega = x/|x|$, cf. [Sauter/Schwab]. They essentially state that u is an outgoing wave. An appropriate (complex-valued) solution space is

$$H^1_H(\Omega^{\text{ext}}) := \Big\{u \in L^1_{\text{loc}}(\overline{\Omega}^{\text{ext}}) : \int_{\Omega^{\text{ext}}} \frac{|u(x)|^2 + |\nabla u(x)|^2}{1+|x|^2} + \left|\frac{\partial u}{\partial \omega} - i\kappa u\right|^2 dx < \infty\Big\}.$$

For more details see [Sauter/Schwab], [McLean, Ch. 9].

3.6.3 Exterior Calderón Identities

In the following, we restrict ourselves to the (homogeneous) exterior Laplace problem in 3D.

From Definition 3.22 and the jump relations (Lemma 3.21), we get that

$$\begin{aligned} \gamma_0^{\text{ext}}\widetilde{V} &= V, & \gamma_0^{\text{ext}}\widetilde{W} &= \sigma I + K, \\ \gamma_1^{\text{ext}}\widetilde{V} &= (-1+\sigma)I + K', & \gamma_1^{\text{ext}}\widetilde{W} &= -D. \end{aligned} \tag{3.38}$$

The representation formula from Theorem 3.36 and the relations (3.38) lead immediately to the exterior Calderón identies

$$\begin{bmatrix} \gamma_0^{\text{ext}} u \\ \gamma_1^{\text{ext}} u \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma I + K & -V \\ -D & (1-\sigma)I - K' \end{bmatrix}}_{=\mathcal{C}^{\text{ext}}} \begin{bmatrix} \gamma_0^{\text{ext}} u \\ \gamma_1^{\text{ext}} u \end{bmatrix}, \tag{3.39}$$

and one can show that $\mathcal{C}^{\text{ext}}$ is a projection.

Remark* 3.38. The same formulae hold for the general differential operator L with $c \neq 0$ for $d = 2$ and $d = 3$. For the Laplacian in 2D, a suitable constant term has to be added to the first line of (3.39) (namely $\langle w_{\text{eq}}, \gamma_0^{\text{ext}} u \rangle_\Gamma$, cf. Def. 3.42 below).

3.6.4 Exterior Boundary Integral Equations

Assume again that $d = 3$ and $L = -\Delta$. Using the first line of (3.39), one can derive the boundary integral equation

$$\text{find } t \in H^{-1/2}(\Gamma): \qquad Vt = [(\sigma-1)I + K]g_D \qquad \text{in } H^{1/2}(\Gamma) \tag{3.40}$$

for the conormal derivative $t := \gamma_0^{\text{ext}} u$ of the solution u of the exterior Dirichlet problem

$$Lu = 0 \quad \text{in } \mathcal{D}^*(\Omega^{\text{ext}}), \qquad \gamma_0^{\text{ext}} u = g_D \quad \text{in } H^{1/2}(\Gamma).$$

If $t \in H^{-1/2}(\Gamma)$ solves (3.40), then

$$u = \widetilde{W} g_D - \widetilde{V} t$$

defines a solution of (3.40) (where the correct solution space has to be used).

The analogous steps lead to the boundary integral equation

$$Dv = -(\sigma I + K')g_N \qquad \text{in } H^{-1/2}(\Gamma)$$

for the exterior Neumann problem.

3.7 Ellipticity of V and D

Recall that the boundary integral equations for the interior/exterior Dirichlet/Neumann problem have the form

$$\text{find } t \in H^{-1/2}(\Gamma): \quad Vt = g_1 \quad \text{in } H^{1/2}(\Gamma), \tag{3.41}$$

$$\text{find } v \in H^{1/2}(\Gamma): \quad Dv = g_2 \quad \text{in } H^{-1/2}(\Gamma), \tag{3.42}$$

where the right-hand side (g_1 or g_2) depends on the given data. Recall that $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $D : H^{1/2} \to H^{-1/2}(\Gamma)$ are self-adjoint operators. Hence, we can write (3.41)–(3.42) in variational form with symmetric bilinear forms:

$$\text{find } t \in H^{-1/2}(\Gamma): \quad \langle \tau, Vt\rangle_\Gamma = \langle \tau, g_1\rangle_\Gamma \quad \forall \tau \in H^{-1/2}(\Gamma), \tag{3.43}$$

$$\text{find } v \in H^{1/2}(\Gamma): \quad \langle Dv, \varphi\rangle_\Gamma = \langle g_2, \varphi\rangle_\Gamma \quad \forall \varphi \in H^{1/2}(\Gamma). \tag{3.44}$$

In this section, we shall investigate if/when V, D are elliptic, or in other words, if/when the bilinear forms associated to V, D are coercive. If they are, the Lax-Milgram theorem will imply the *well-posedness* of (3.41), (3.42).

3.7.1 Ellipticity of V

Definition 3.39 ($H_*^{-1/2}(\Gamma)$)**.** We define

$$H_*^{-1/2}(\Gamma) := \{w \in H^{-1/2}(\Gamma) : \langle w, 1\rangle_\Gamma = 0\}.$$

Theorem 3.40 (ellipticity of V)**.** *Let $L = -\Delta$. Then there exists a constant $c_V > 0$ such that*

$$\langle w, Vw\rangle_\Gamma \geq c_V \|w\|^2_{H^{-1/2}(\Gamma)} \qquad \begin{cases} \forall w \in H_*^{-1/2}(\Gamma) & \text{if } d = 2, \\ \forall w \in H^{-1/2}(\Gamma) & \text{if } d = 3. \end{cases}$$

Proof. Assume that $w \in H^{-1/2}(\Gamma)$ if $d = 3$, and $w \in H_*^{-1/2}(\Gamma)$ if $d = 2$. We define $u := \widetilde{V}w$. From Theorem 3.18, we see that $u_{|\Omega^{\text{int}}} \in H^1(\Omega^{\text{int}})$. From Theorem 3.35, we see that $u_{|\Omega^{\text{ext}}} \in H^1_E(\Omega^{\text{ext}})$.

Since $-\Delta u = 0$ in $\Omega^{\text{int/ext}}$, we obtain from Def. 2.36 and (3.36)

$$\int_{\Omega^{\text{int}}} \nabla u \cdot \nabla v \, dx = \langle \gamma_1^{\text{int}} u, \gamma_0^{\text{int}} v\rangle_\Gamma \qquad \forall v \in H^1(\Omega^{\text{int}}),$$

$$\int_{\Omega^{\text{ext}}} \nabla u \cdot \nabla v \, dx = -\langle \gamma_1^{\text{ext}} u, \gamma_0^{\text{ext}} v\rangle_\Gamma \qquad \forall v \in H^1_E(\Omega^{\text{ext}}).$$

Choosing $v = u$ and using the jump relations (Lemma 3.21), we obtain

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \;=\; \langle \underbrace{\gamma_1^{\rm int} u - \gamma_1^{\rm ext} u}_{= -[\![\gamma_1 u]\!] = w}, \quad \underbrace{\gamma_0 u}_{= \gamma_0 \widetilde{V} w = V w} \rangle_\Gamma \;=\; \langle w, \, V w \rangle_\Gamma \,. \tag{3.45}$$

The expression on the left-hand side is usually referred to the *total energy* (of the potential $\widetilde{V}w$). For the special case of the Laplacian, we can adapt the proof of Theorem (2.36) and show that

$$\|\gamma_1^{\rm int} u\|_{H^{-1/2}(\Gamma)} \;\le\; C_{\rm IT}^{\rm int} \Big(\int_{\Omega^{\rm int}} |\nabla u| \, dx \Big)^{1/2} \tag{3.46}$$

$$\|\gamma_1^{\rm ext} u\|_{H^{-1/2}(\Gamma)} \;\le\; C_{\rm IT}^{\rm ext} \Big(\int_{\Omega^{\rm ext}} |\nabla u| \, dx \Big)^{1/2}, \tag{3.47}$$

where $C_{\rm IT}^{\rm int}$, $C_{\rm IT}^{\rm ext}$ are the constants of the interior/exterior *inverse* trace theorem. Combining (3.45) and (3.46) yields

$$\begin{aligned} \langle w, \, V w \rangle_\Gamma \;&=\; \int_{\Omega^{\rm int}} |\nabla u|^2 \, dx + \int_{\Omega^{\rm ext}} |\nabla u|^2 \, dx \\ &\ge\; (C_{\rm IT}^{\rm int})^{-2} \, \|\gamma_1^{\rm int} u\|_{H^{-1/2}(\Gamma)}^2 + (C_{\rm IT}^{\rm ext})^{-2} \, \|\gamma_1^{\rm ext} u\|_{H^{-1/2}(\Gamma)}^2 \\ &\ge\; \min\{(C_{\rm IT}^{\rm int})^{-2}, \, (C_{\rm IT}^{\rm ext})^{-2}\} \, \Big(\|\gamma_1^{\rm int} u\|_{H^{-1/2}(\Gamma)}^2 + \|\gamma_1^{\rm ext} u\|_{H^{-1/2}(\Gamma)}^2 \Big). \end{aligned} \tag{3.48}$$

From the jump relations and (3.48), we conclude that

$$\|w\|_{H^{-1/2}(\Gamma)}^2 \;=\; \|\gamma_1^{\rm int} u - \gamma_1^{\rm ext} u\|_{H^{-1/2}(\Gamma)}^2 \;\le\; 2 \, \Big(\|\gamma_1^{\rm int} u\|_{H^{-1/2}(\Gamma)}^2 + \|\gamma_1^{\rm ext} u\|_{H^{-1/2}(\Gamma)}^2 \Big).$$

This implies $\langle w, \, V w \rangle_\Gamma \ge c_V \, \|w\|_{H^{-1/2}(\Gamma)}^2$ with $c_V = \frac{1}{2} \, \min\{(C_{\rm IT}^{\rm int})^{-2}, \, (C_{\rm IT}^{\rm ext})^{-2}\}$. □

Exercise 22. Prove the first line of (3.46) for all $u \in H^1(\Omega^{\rm int})$ with $\Delta u = 0$ in $\mathcal{D}^*(\Omega^{\rm int})$. *Hint:* use Step 1 in the proof of Thm. 2.36 to define the functional $\gamma_1^{\rm int} u$.

Theorem 3.40 implies that V is $H^{-1/2}(\Gamma)$-elliptic in 3D and at least $H_*^{-1/2}(\Gamma)$-elliptic in 2D. In order to see what happens on the "rest" in 2D, we first observe that

$$H^{-1/2}(\Gamma) \;=\; H_*^{-1/2}(\Gamma) \;\oplus\; \operatorname{span}\{1_\Gamma\}, \tag{3.49}$$

where 1_Γ denotes the constant function on Γ with value one. Moreover, each $w \in H^{-1/2}(\Gamma)$ can be decomposed *uniquely*: $w = w_1 + w_0$ with

$$w_1 \;=\; w - \frac{1}{|\Gamma|} \langle w, 1 \rangle_\Gamma \;\in\; H_*^{-1/2}(\Gamma), \qquad w_0 \;=\; \frac{1}{|\Gamma|} \langle w, 1 \rangle_\Gamma \;=\; \text{const},$$

where $|\Gamma| = \langle 1, \, 1 \rangle_\Gamma = (1, \, 1)_{L^2(\Gamma)} = \int_\Gamma \, ds$.

Lemma 3.41 (equilibrium density)**.** *For $L = -\Delta$, there exists a unique distribution $w_{\rm eq} \in H^{-1/2}(\Gamma)$ such that $Vw_{\rm eq} = \text{const}$ and*

$$\left.\begin{aligned} \langle w, Vw_{\rm eq}\rangle &= 0 \qquad \forall w \in H_*^{-1/2}(\Gamma), \\ \langle w_{\rm eq}, 1\rangle &= 1. \end{aligned}\right\} \tag{3.50}$$

Exercise 23. Prove Lemma 3.41. *Hint:* Step 1. Decompose $w_{\rm eq} = w_0 + w_1$ as above. From (3.50), compute w_0, and derive a variational equation for w_1. Use Lax-Milgram to show existence and uniqueness of $w_1 \in H_*^{-1/2}(\Gamma)$. Step 2. Using (3.49), show that there exists $\lambda \in \mathbb{R}$ (compute its value!) such that

$$\langle w, Vw_{\rm eq}\rangle + \lambda\langle w, 1\rangle_\Gamma = 0 \qquad \forall w \in H^{-1/2}(\Gamma).$$

and conclude from this that $Vw_{\rm eq}$ is constant.

Definition 3.42. The distribution $w_{\rm eq}$ from Lemma 3.41 is called *equilibrium density.*

Apparently, $\langle w_{\rm eq}, \underbrace{Vw_{\rm eq}}_{=\text{const}}\rangle = (Vw_{\rm eq})\ \underbrace{\langle w_{\rm eq}\, 1\rangle}_{=1}$.

Lemma 3.43 (without proof)**.** *Let $L = -\Delta$ and $d = 2$. If $\operatorname{diam}(\Omega^{\rm int}) < 2$, then $\langle w_{\rm eq}, Vw_{\rm eq}\rangle = Vw_{\rm eq} > 0$.*

The source of the condition is the estimate $Vw_{\rm eq} \geq -\frac{1}{2\pi}\log(\frac{1}{2}\operatorname{diam}(\Omega^{\rm int}))$.

Remark* 3.44. For $d = 2$, the value $Vw_{\rm eq}$ depends on the diameter of $\Omega^{\rm int}$, because we have chosen the particular fundamental solution $G(x) = -\frac{1}{2\pi}\log|x|$. If we choose $G_r(x) = \frac{1}{2\pi}\log(r/|x|)$, and call the associated operator V_r, then for all $w \in H_*^{-1/2}(\Gamma)$, $V_r w = Vw$. The equilibrium density can be shown to be independent of $r > 0$, but the value $V_r w_{\rm eq}$ can take any value in $(-\infty, +\infty)$; in particular there exists a number r such that $V_r w_{\rm eq} = 0$. The quantity $e^{-2\pi V w_{\rm eq}}$ is called *logarithmic capacity* and has been well-studied in potential theory and harmonic analysis.

Lemma 3.45. *Let $L = -\Delta$, $d = 2$, and assume that $\operatorname{diam}(\Omega^{\rm int}) < 2$. Then there exists a constant $\widetilde{c}_V > 0$ such that*

$$\langle w, Vw\rangle \geq \widetilde{c}_V \|w\|^2_{H^{-1/2}(\Gamma)} \qquad \forall w \in H^{-1/2}(\Gamma).$$

Proof. Let $w \in H^{-1/2}(\Gamma)$ be arbitrary but fixed. There exists a unique decomposition of $w = \widetilde{w} + \beta w_{\rm eq}$ with $\widetilde{w} \in H_*^{-1/2}(\Gamma)$ and $\beta \in \mathbb{R}$. Due to (3.50), this decomposition is "V-orthogonal":

$$\langle \widetilde{w}, Vw_{\rm eq}\rangle = 0.$$

Hence,

$$\begin{aligned}\langle w,\, Vw\rangle &= \langle \widetilde{w},\, V\widetilde{w}\rangle + \beta^2\langle w_{\rm eq},\, Vw_{\rm eq}\rangle \\ &\overset{\text{Thm. 3.40}}{\geq} c_V\,\|\widetilde{w}\|^2_{H^{-1/2}(\Gamma)} + (Vw_{\rm eq})\,\beta^2 \\ &\geq \min\Big(c_V\,\frac{Vw_{\rm eq}}{\|w_{\rm eq}\|^2_{H^{-1/2}(\Gamma)}}\Big)\Big[\|\widetilde{w}\|^2_{H^{-1/2}(\Gamma)} + \beta^2\,\|w_{\rm eq}\|^2_{H^{-1/2}(\Gamma)}\Big].\end{aligned}$$

An elementary argument yields

$$\|w\|^2_{H^{-1/2}(\Gamma)} \;\leq\; 2(\|\widetilde{w}\|^2_{H^{-1/2}(\Gamma)} + \beta^2\|w_{\rm eq}\|^2_{H^{-1/2}(\Gamma)}).$$

Hence, $\langle w,\, Vw\rangle \geq \widetilde{c}_V\,\|w\|^2_{H^{-1/2}(\Gamma)}$ with $\widetilde{c}_V = \min\big(c_V,\, \frac{Vw_{\rm eq}}{\|w_{\rm eq}\|^2_{H^{-1/2}(\Gamma)}}\big) > 0$. □

Summary: Let $L = -\Delta$. If $d = 3$, or if $d = 2$ and $\operatorname{diam}(\Omega^{\rm int}) < 2$, then V is elliptic on the whole of $H^{-1/2}(\Gamma)$. In that case, the inverse $V^{-1} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ exists and is bounded, elliptic, and self-adjoint.

Remark 3.46. In practice, the (sufficient) condition $\operatorname{diam}(\Omega^{\rm int}) < 2$ for $L = -\Delta$, $d = 2$ can be enforced easily by a scaling of the coordinates.

Remark* 3.47. If $Lu = -\operatorname{div}(A\nabla u) + c\,u$ with $c > 0$, then V is $H^{-1/2}(\Gamma)$-elliptic (without any restrictions). For $c < 0$, V is Fredholm with index zero, which implies that V is coercive with respect to the pivot space $L^2(\Gamma)$.

3.7.2 Ellipticity of D

Definition 3.48 ($H^{1/2}_*(\Gamma)$)**.** Let $L = -\Delta$. We define

$$H^{1/2}_*(\Gamma) \;:=\; \{v \in H^{1/2}(\Gamma) : \langle w_{\rm eq},\, v\rangle_\Gamma = 0\}.$$

Apparently, we have the unique decomposition

$$H^{1/2}(\Gamma) \;=\; H^{1/2}_*(\Gamma) \;\oplus\; \operatorname{span}\{1_\Gamma\}.$$

Exercise 24. Show that V maps $H^{-1/2}_*(\Gamma)$ to $H^{1/2}_*(\Gamma)$ and is an isomorphism between these spaces.

Theorem 3.49. *Let $L = -\Delta$. Then there exist positive constants c_D, $\widetilde{c}_D > 0$ such that*

$$\langle Dv,\, v\rangle \;\geq\; c_D\,|v|^2_{H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}(\Gamma), \tag{3.51}$$

$$\langle Dv,\, v\rangle \;\geq\; \widetilde{c}_D\,\|v\|^2_{H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}_*(\Gamma), \tag{3.52}$$

and $\ker(D) = \operatorname{span}\{1_\Gamma\}$.

Proof. Step 1. The proof of (3.51) is similar to that of Theorem 3.40. We set $u = \widetilde{W}v$. This time, $u_{|\Omega^{\text{ext}}} \in H^1_E(\Omega^{\text{ext}})$. From the jump relations and Green's identity, we obtain

$$\begin{aligned}\langle \underbrace{Dv}_{=-\gamma_1\widetilde{W}v}, v\rangle &= \langle -\gamma_1 u, [\![\gamma_0 u]\!]\}\rangle = -\langle \gamma_1^{\text{ext}} u, \gamma_0^{\text{ext}} u\rangle + \langle \gamma_1^{\text{int}} u, \gamma_0^{\text{int}} u\rangle \\ &= \int_{\Omega^{\text{ext}}} |\nabla u|^2\, dx + \int_{\Omega^{\text{int}}} |\nabla u|^2\, dx.\end{aligned}$$

Using the interior trace theorem and Poincaré's inequality, we get

$$\begin{aligned}|\gamma_0^{\text{int}} u|^2_{H^{1/2}(\Gamma)} &= |\gamma_0^{\text{int}} u - \overline{u}^{\Omega^{\text{int}}}|^2_{H^{1/2}(\Gamma)} \leq (C_{\text{tr}}^{\text{int}})^2 \, \|u - \overline{u}^{\Omega^{\text{int}}}\|^2_{H^1(\Omega^{\text{int}})} \\ &\leq (C_{\text{tr}}^{\text{int}})^2 \, (1 + C_P(\Omega^{\text{int}})^2) \, |u|^2_{H^1(\Omega^{\text{int}})},\end{aligned}$$

i.e. the interior trace inequality holds with the respective seminorms. With similar (but not completely analogous) techniques, one can show that

$$|\gamma_0^{\text{ext}} u|^2_{H^{1/2}(\Gamma)} \leq C\, |u|^2_{H^1(\Omega^{\text{ext}})}$$

for some constant $C \in (0, \infty)$. Combination of the last three results yields

$$\langle Dv, v\rangle \geq 2\, c_D \left(|\gamma_0^{\text{ext}} u|^2_{H^{1/2}(\Gamma)} + |\gamma_0^{\text{int}} u|^2_{H^{1/2}(\Gamma)}\right) \geq c_D\, |\underbrace{\gamma_0^{\text{ext}} u - \gamma_0^{\text{int}} u}_{=v}|^2_{H^{1/2}(\Gamma)},$$

where $2c_D = \min((C_{\text{tr}}^{\text{int}})^2(1 + C_P(\Omega^{\text{int}})^2)^{-1}, C^{-1}) > 0$.

Step 2. We show that $\ker(D) = \operatorname{span}\{1_\Gamma\}$. Apparently, the constant function $1_{\Omega^{\text{int}}}$ solves the homogeneous Laplace equation in Ω^{int}, with Dirichlet datum 1_Γ and $\gamma_1^{\text{int}} 1_{\Omega^{\text{int}}} = 0$. Consequently, the representation formula yields

$$1_{\Omega^{\text{int}}} = -\widetilde{W} 1_\Gamma + 0,$$

and thus, $0 = -\gamma_1^{\text{int}} \widetilde{W} 1_\Gamma = D 1_\Gamma$, which implies $\operatorname{span}(1_\Gamma) \subseteq \ker(D)$. However, since $\operatorname{span}(1_\Gamma)$ is the kernel of the seminorm $|\cdot|_{H^{1/2}(\Gamma)}$ and because of (3.51), the kernel cannot be larger.

Step 3. Recall that $H^{1/2}_*(\Gamma)$ has codimension 1 with respect to $H^{1/2}(\Gamma)$. Using e.g., Sobolev's norm theorem, one can show the Poincaré-type inequality

$$\|v\|_{L^2(\Gamma)} \leq C\, |v|_{H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}_*(\Gamma).$$

With this inequality, (3.52) can be derived straightforwardly from (3.51). □

Remark* 3.50. If $Lu = -\operatorname{div}(A\nabla u) + cu$ with $c > 0$, then D is elliptic on the whole of $H^{1/2}(\Gamma)$.

3.7.3 Solvability of Two Boundary Integral Equations

Single Layer Boundary Integral Equation Assume that $L = -\Delta$ and that either $d = 3$, or $d = 2$ and $\mathrm{diam}(\Omega^{\mathrm{int}}) < 2$. Then the boundary integral equation

$$\text{find } t \in H^{-1/2}(\Gamma): \qquad Vt \;=\; g_1 \qquad \text{in } H^{1/2}(\Gamma)$$

admits a unique solution due to Lax-Milgram.

Hypersingular Boundary Integral Equation Assume that $L = -\Delta$. Then the boundary integral equation

$$\text{find } v \in H^{1/2}(\Gamma): \qquad D\,v \;=\; g_2 \qquad \text{in } H^{-1/2}(\Gamma)$$

is solvable if and only if $g_2 \in H_*^{-1/2}(\Gamma)$. The solution v is unique up to an additive constant.

Exercise 25. Show the statement above.

In case of the interior Neumann problem, $g_2 = (\frac{1}{2}I - K')g_N$. As one can show (see Sect. 3.7.4 below), the usual compatibility condition $\langle g_N,\, 1\rangle_\Gamma = 0$ implies $g_2 \in H_*^{-1/2}(\Gamma)$.

3.7.4 Properties of K and K' *

Let $L = -\Delta$. Then the operators $\frac{1}{2}I \pm K$ are isomorphisms between $H_*^{1/2}(\Gamma)$ and $H_*^{1/2}(\Gamma)$, and $\frac{1}{2}I \pm K'$ are isomorphisms between $H_*^{-1/2}(\Gamma)$ and $H_*^{-1/2}(\Gamma)$. Furthermore,

$$\begin{aligned} &\ker(\tfrac{1}{2}I + K) = \mathrm{span}(1_\Gamma), \qquad &&\ker(\tfrac{1}{2}I + K') = \mathrm{span}(w_{\mathrm{eq}}),\\ &(\tfrac{1}{2}I - K)1_\Gamma = 1_\Gamma\,, &&(\tfrac{1}{2}I - K')w_{\mathrm{eq}} = w_{\mathrm{eq}}. \end{aligned}$$

Furthermore, there exists a *contraction constant* $c_K \in (\frac{1}{2},\, 1)$ such that

$$(1 - c_K)\|v\|_{V^{-1}} \;\le\; \|(\tfrac{1}{2}I \pm K)v\|_{V^{-1}} \;\le\; c_K\,\|v\|_{V^{-1}} \qquad \forall v \in H_*^{1/2}(\Gamma),$$

where V^{-1} denotes the inverse of $V_{|H_*^{-1/2}(\Gamma)}$ and $\|v\|_{V^{-1}} = \sqrt{\langle V^{-1}v,\, v\rangle}$. The contraction property can, e.g., be used to show existence and uniqueness of the 2^{nd} kind integral equations from Sect. 3.5.5 via Banach's fixed point theorem.

If the boundary Γ is smooth, then K, K' are even compact operators, however this property is lost for general Lipschitz domains.

3.8 Steklov-Poincaré Operators*

Let $L = -\Delta$ and assume, for simplicity that $d = 3$. Let $S^{\text{int}} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ denote the interior Steklov-Poincaré operator from Sect. 3.5.2. Recall the interior Calderón identities (3.21) for the homogenoeus PDE:

$$\begin{bmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{bmatrix} \begin{bmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{bmatrix}.$$

Since $\gamma_1^{\text{int}} u = S^{\text{int}} \gamma_0^{\text{int}} u$ and because V is invertible, we obtain the two alternative representations

$$S^{\text{int}} = V^{-1}(\tfrac{1}{2}I + K) = D + (\tfrac{1}{2}I + K')V^{-1}(\tfrac{1}{2}I + K), \tag{3.53}$$

where the second one reflects (again) the self-adjointness of S^{int}. Based on Theorem 3.49, one can show that S^{int} is elliptic with respect to the $H^{1/2}(\Gamma)$-seminorm.

Analogously, one can show the representation

$$-S^{\text{ext}} = V^{-1}(\tfrac{1}{2}I - K) = D + (\tfrac{1}{2}I - K')V^{-1}(\tfrac{1}{2}I - K) \tag{3.54}$$

for the exterior Steklov-Poincaré operator. The operator $-S^{\text{ext}}$ is $H^{1/2}(\Gamma)$-elliptic. Furthermore,

$$V^{-1} = S^{\text{int}} - S^{\text{ext}}.$$

Chapter 4

Galerkin BEM

In this chapter, we discuss how to construct a Galerkin boundary element method based on a surface triangulation of the boundary Γ. For the specific case of two boundary integral equations for the Laplacian, we provide a priori error estimates. Finally, we discuss some properties of the boundary element matrices.

4.1 Construction of Galerkin BEM

4.1.1 Surface Triangulations

Throughout this chapter, we assume that Γ splits into disjoint parts $\{\Gamma_j\}_{j=1}^N$,

$$\Gamma \;=\; \bigcup_{j=1}^{N} \overline{\Gamma}_j, \qquad \Gamma_j \cap \Gamma_k = \emptyset \quad \text{for } j \neq k,$$

where each part Γ_j is the image of a $(d-1)$-dimensional parameter interval/polygon $\mathcal{Q}_j$ under a diffeomorphism ζ_j (bijective, ζ_j, $\zeta_j^{-1} \in C^1$). We now consider standard triangulations of the parameter domains $\mathcal{Q}_j$:

- for $d = 2$, $\mathcal{Q}_j$ splits into simple intervals,
- for $d = 3$, $\mathcal{Q}_j$ splits into triangles.

Let τ_i denote the *images* of these elements under the respective mapping ζ_j, such that

$$\Gamma \;=\; \bigcup_{i=1}^{n_h^{\mathrm{el}}} \overline{\tau}_i \,,$$

where n_h^{el} denotes the total number of elements. The resulting (global) mesh/triangulation of Γ is denoted by $\mathcal{T}_h(\Gamma) = \{\tau_i\}_{i=1}^{n_h^{\mathrm{el}}}$.

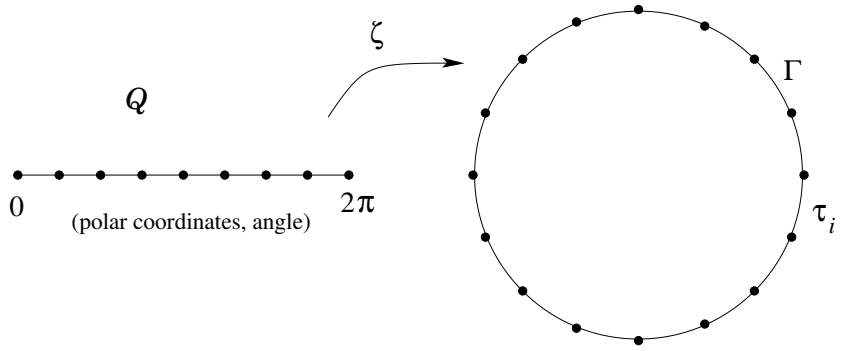


Figure 4.1: Illustration of a meshing a circle, Example 4.1 (b). *Left:* Mesh of the 1D parameter domain $\mathcal{Q}$. *Right:* Resulting surface mesh of Γ.

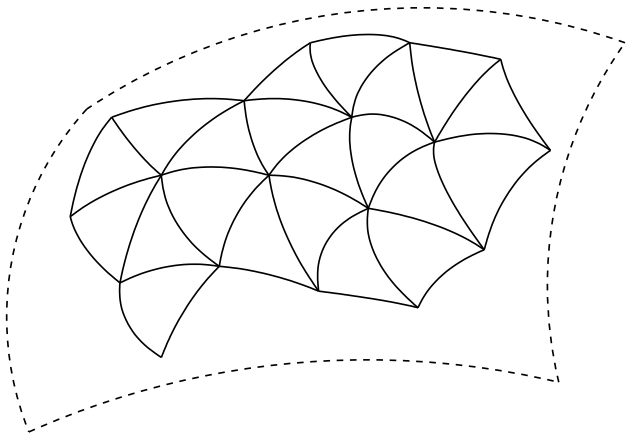


Figure 4.2: Illustration of surface mesh in 3D (slightly exaggerated).

Example 4.1. (a) If Γ_j are already a straight lines ($d = 2$) or flat faces ($d = 3$), then τ_i are straight line segments or flat triangles, respectively.

(b) Let Γ be the circle $\partial B_1(0) \subset \mathbb{R}^2$. Using polar coordinates, we transform the interval $\mathcal{Q} := [0, 2\pi)$ to Γ with $\zeta(\varphi) := (\cos(\varphi), \sin(\varphi))$. We subdivide $\mathcal{Q}$ into subintervals $(\varphi_k, \varphi_{k+1})$, e.g., $\varphi_k := 2\pi k/n_h^{\mathrm{el}}$, $k = 0, \ldots, n_h^{\mathrm{el}}$. The resulting surface mesh $\mathcal{T}_h(\Gamma)$ consisting of n_h^{el} arcs is illustrated in Figure 4.1

(c) For an illustration of a 3D mesh consisting of curved triangles, see Figure 4.2.

Definition 4.2. 1. The global mesh $\mathcal{T}_h(\Gamma)$ is called *conforming* if the intersection of two different elements τ_i, τ_j is either empty, or a vertex of both elements, or (for $d = 3$) a (possibly curved) edge of both elements. (By a vertex/edge we mean the image of the corresponding parameter vertex/edge) In particular, hanging nodes are not allowed.

2. We define the *local* and *global mesh size* by

$$h_i := |\tau_i|^{1/(d-1)}, \qquad h := \max_{i=1}^{n_h^{\mathrm{el}}} h_i \,.$$

3. A family $\{\mathcal{T}_h(\Gamma)\}_h$ of surface meshes is called *shape regular* if

   - the corresponding families of parameter meshes are *shape regular*, i.e., $\widetilde{\rho}_i \geq c\,\widetilde{h}_i$, where $\widetilde{h}_i$ is the diameter of the corresponding parameter element and $\widetilde{\rho}_i$ the radius of the largest inscribed ball in the parameter element, with uniform constant $c > 0$, and if
   - the geometry mappings ζ_j, ζ_j^{-1} have uniformly bounded gradients and determinants.

   In that case, $\operatorname{diam}(\tau_i) \simeq h_i \simeq \widetilde{h}_i \simeq \widetilde{\rho}_i$.

4. A family $\{\mathcal{T}_h(\Gamma)\}_h$ of surface meshes is called *quasi uniform* if it is shape regular and if there exists a uniform constant $c > 0$ such that

$$h_i \geq c\,h \qquad \forall i = 1, \ldots, n_h^{\mathrm{el}} \,.$$

In the following, we assume that $\mathcal{T}_h(\Gamma)$ is conforming and shape regular.

4.1.2 Trial Spaces

Let $\mathcal{T}_h(\Gamma_j) := \{\tau \in \mathcal{T}_h(\Gamma) : \tau \subset \Gamma_j\}$ denote the restriction of $\mathcal{T}_h(\Gamma)$ to Γ_j and let $\mathcal{P}^k$ denote the space of ($(d-1)$-variate) polynomials of degree $\leq k$. We define the spaces

$$\begin{aligned} S_h^0(\Gamma) &:= \big\{w \in L^2(\Gamma) : w_{|\tau} \circ \zeta_j \in \mathcal{P}^0 \quad \forall \tau \in \mathcal{T}_h(\Gamma_j), \quad \forall j = 1, \ldots, N\big\}, \\ S_h^1(\Gamma) &:= \big\{v \in C(\Gamma) : v_{|\tau} \circ \zeta_j \in \mathcal{P}^1 \quad \forall \tau \in \mathcal{T}_h(\Gamma_j), \quad \forall j = 1, \ldots, N\big\}. \end{aligned}$$

The functions in $S_h^0(\Gamma)$ are *piecewise constant* with respect to $\mathcal{T}_h(\Gamma)$ and typically *discontinuous*. The functions in $S_h^1(\Gamma)$ are *continuous* and may informally be called *piecewise linear*.

Lemma 4.3. *The trial spaces above fulfill*

$$S_h^0(\Gamma) \subset H^{-1/2}(\Gamma), \qquad S_h^1(\Gamma) \subset H^{1/2}(\Gamma),$$

i.e., they are conforming to the spaces involved in Problems (3.41) *and* (3.42).

A natural basis of $S_h^0(\Gamma)$ is given by $\{\varphi_i^0\}_{i=1}^{n_h^{\mathrm{el}}}$, where

$$\varphi_i^0(x) \;=\; \begin{cases} 1 & \text{if } x \in \tau_i \\ 0 & \text{else.} \end{cases}$$

Let x_k denote the *vertices (nodes)* of $\mathcal{T}_h(\Gamma)$ and n_h^{vert} the total number of vertices. Then a natural basis of $S_h^1(\Gamma)$ is given by $\{\varphi_k^1\}_{k=1}^{n_h^{\mathrm{vert}}}$,where

$$\varphi_k^1 \in S_h^1(\Gamma), \qquad \varphi_k^1(x_\ell) \;=\; \delta_{k\ell}\,.$$

4.1.3 Galerkin BEM

Recall the boundary integral equations for the homogeneous interior Dirichlet and Neumann problem:

$$\text{find } t \in H^{-1/2}(\Gamma): \quad \langle \tau,\, Vt\rangle_\Gamma = \langle \tau,\, (\tfrac{1}{2}I - K)g_D\rangle_\Gamma \quad \forall \tau \in H^{-1/2}(\Gamma), \tag{4.1}$$

$$\text{find } v \in H^{-1/2}(\Gamma): \langle Dv,\, \varphi\rangle_\Gamma = \langle (\tfrac{1}{2}I - K')g_N,\, \varphi\rangle_\Gamma \quad \forall \varphi \in H^{1/2}(\Gamma). \tag{4.2}$$

The corresponding Galerkin formulations read

$$\text{find } t_h \in S_h^0(\Gamma): \quad \langle \tau_h,\, Vt_h\rangle_\Gamma = \langle \tau_h,\, (\tfrac{1}{2}I - K)g_D\rangle_\Gamma \quad \forall \tau_h \in S_h^0(\Gamma), \tag{4.3}$$

$$\text{find } v_h \in S_h^1(\Gamma): \langle Dv_h,\, \varphi_h\rangle_\Gamma = \langle (\tfrac{1}{2}I - K')g_N,\, \varphi_h\rangle_\Gamma \quad \forall \varphi_h \in S_h^1(\Gamma). \tag{4.4}$$

For the other boundary integral equations involving V or D, one can of course proceed analogously.

4.1.4 Matrix-Vector Form

For simplicity, assume that the data of (4.1), (4.2) fulfills $g_D \in S_h^1(\Gamma)$ and $g_N \in S_h^0(\Gamma)$, and let

$$\underline{g}_{D,h} \in \mathbb{R}^{n_h^{\mathrm{vert}}}, \qquad \underline{g}_{N,h} \in \mathbb{R}^{n_h^{\mathrm{el}}}$$

denote the coefficient vectors with respect to the natural bases.

We define the matrix

$$\begin{aligned}[V_h]_{ij} &:= \langle \varphi_j^0,\, V\varphi_i^0\rangle_\Gamma \qquad \text{for } i,\, j = 1,\dots,n_h^{\mathrm{el}} \\ &= \int_{\tau_j}\int_{\tau_i} G(x-y)\, ds_y,\, ds_x\,,\end{aligned}$$

where for the second line, we have used Lemma 3.24 and the fact that φ_i^0, $\varphi_j^0 \in L^\infty(\Gamma)$. Moreover, we define the matrices

$$\left.\begin{aligned}[K_h]_{ki} &:= \langle \varphi_i^0,\, K\varphi_k^1\rangle_\Gamma \\ [M_h]_{ki} &:= \textstyle\int_\Gamma \varphi_i^0\, \varphi_k^1\, ds\end{aligned}\right\} \quad \text{for } i = 1,\dots,n_h^{\mathrm{el}},\quad k = 1,\dots,n_h^{\mathrm{vert}}\,,$$

and

$$[D_h]_{k\ell} := \langle D\varphi_\ell^1,\, \varphi_k\rangle_\Gamma \qquad \text{for } k,\, \ell = 1,\dots,n_h^{\mathrm{vert}}\,.$$

With these notations, (4.3), (4.4) are equivalent to

$$\text{find } \underline{t}_h \in \mathbb{R}^{n_h^{\mathrm{el}}}: \quad V_h\,\underline{t}_h = (\tfrac{1}{2}M_h - K_h)\underline{g}_{D,h}\,, \tag{4.5}$$

$$\text{find } \underline{v}_h \in \mathbb{R}^{n_h^{\mathrm{vert}}}: D_h\,\underline{v}_h = (\tfrac{1}{2}M_h^\top - K_h^\top)\underline{g}_{N,h}\,, \tag{4.6}$$

in the sense that the coefficient vectors $\underline{t}_h$, $\underline{v}_h$ correspond to t_h, v_h, resp.

Calculation of matrix entries will be discussed briefly in Sect. 4.3 below. The conditioning of V_h and D_h is the subject of Sect. 4.4. Sufficient storage and application of boundary element matrices is treated in Chapter 5.

4.2 A Priori Error Estimates

In this section, we provide with a priori estimates for the errors $t - t_h$ and $v - v_h$ of the Dirichlet and Neumann problem, (4.1)–(4.4). Under the assumption that the exact solution (t or v) is sufficiently smooth, we shall obtain convergence as the mesh parameter $h \to 0$.

4.2.1 The Dirichlet Problem

Assume that the single layer potential operator V is elliptic (cf. Sect. 3.7). From Céa's lemma, it follows immediately that the solutions $t \in H^{-1/2}(\Gamma)$ and $t_h \in S_h^0(\Gamma)$ of (4.1) resp. (4.3) fulfill

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \le C \inf_{w_h \in S_h^0(\Gamma)} \|t - w_h\|_{H^{-1/2}(\Gamma)}\,, \tag{4.7}$$

where C is the reciprocal of the constant of boundedness and ellipticity of V with respect to the norm $\|\cdot\|_{H^{-1/2}(\Gamma)}$.

Similarly to standard FEM, we aim at an error bound of the form $C\,h^\alpha$ under the assumption that t is smoother than $H^{-1/2}(\Gamma)$. Recall, however, that t is the normal derivative of the solution u. Since Γ is only piecewise smooth and the normal vector is discontinuous, we cannot expect $t \in H^s(\Gamma)$ with $s > 0$.

Piecewise Sobolev Spaces

Definition 4.4. Following [Steinbach], we set

$$H^s_{\mathrm{pw}}(\Gamma) := \{v \in L^2(\Gamma) : v_{|\Gamma_j} \in H^s(\Gamma_j)\} \quad \text{for } s \in [0,\,1],$$
$$H^{-s}_{\mathrm{pw}}(\Gamma) := \prod_{j=1}^{J} \underbrace{\widetilde{H}^{-s}(\Gamma_j)}_{=H^s(\Gamma_j)^*} \qquad \text{for } s \in (0,\,1],$$

equipped with norms

$$\|v\|_{H^s_{\mathrm{pw}}(\Gamma)} := \Big(\sum_{j=1}^{J} \|v\|^2_{H^s(\Gamma_j)}\Big)^{1/2}, \qquad \|w\|_{H^{-s}_{\mathrm{pw}}(\Gamma)} := \sum_{j=1}^{J} \|w_j\|_{\widetilde{H}^{-s}(\Gamma_j)}.$$

Lemma 4.5. *Let $w \in H^{-s}_{\mathrm{pw}}(\Gamma)$ with $s > 0$ and define the linear functional $\bar{w}$ by $\langle \bar{w},\, v\rangle_\Gamma := \sum_{j=1}^{N} \langle w_j,\, v_{|\Gamma_j}\rangle_{\Gamma_j}$. Then $\bar{w} \in H^{-s}(\Gamma)$ and*

$$\|\bar{w}\|_{H^{-s}(\Gamma)} \;\le\; \|w\|_{H^{-s}_{\mathrm{pw}}(\Gamma)}.$$

If we identify $\bar{w}$ and w (which is justified), we can say that the embedding $H^{-s}_{\mathrm{pw}}(\Gamma) \subset H^{-s}(\Gamma)$ is continuous (whereas the flipped embedding $H^s(\Gamma) \subset H^s_{\mathrm{pw}}(\Gamma)$ is continuous for $s > 0$).

Proof.* By definition of the dual norm and of $\bar{w}$,

$$\begin{aligned}
\|\bar{w}\|_{H^{-s}(\Gamma)} &= \sup_{v\in H^s(\Gamma)\setminus\{0\}} \frac{|\langle \bar{w},\, v\rangle|}{\|v\|_{H^s(\Gamma)}} \;\le\; \sup_{v\in H^s(\Gamma)\setminus\{0\}} \sum_{j=1}^{J} \frac{|\langle w_j,\, v_{|\Gamma_j}\rangle|}{\|v\|_{H^s(\Gamma)}} \\
&\le \sup_{v\in H^s(\Gamma)\setminus\{0\}} \sum_{j=1}^{J} \frac{|\langle w_j,\, v_{|\Gamma_j}\rangle|}{\|v_{|\Gamma_j}\|_{H^s(\Gamma_j)}} \\
&\le \sum_{j=1}^{J} \sup_{v_j\in H^s(\Gamma_j)\setminus\{0\}} \frac{|\langle w_j,\, v_j\rangle|}{\|v_{|\Gamma_j}\|_{H^s(\Gamma_j)}} \;=\; \|w\|_{H^{-s}_{\mathrm{pw}}(\Gamma)}. \qquad \square
\end{aligned}$$

Exercise 26 (for doctoral students)**.** Show that

$$H^s_{\mathrm{pw}}(\Gamma) \;\subset\; H^{-1/2}_{\mathrm{pw}}(\Gamma) \qquad \text{for } s \in [-\tfrac{1}{2},\,1],$$

and that the embedding is continuos.

An Approximation Result for S_h^0

We study the approximation qualities of S_h^0 on each boundary part Γ_j separately.

Theorem 4.6 (approximation by S_h^0)**.** *Let $\Gamma_j \subset \Gamma$ be a C^1-manifold (as in Sect. 4.1.1). Then, for any Sobolev indices $\sigma \in [-1, 0]$ and $s \in [\sigma, 1]$, there exists a constant C independent of h such that*

$$\inf_{w_h \in S_h^0(\Gamma_j)} \|w - w_h\|_{H^\sigma(\Gamma_j)} \;\leq\; C\, h^{s-\sigma}\, \|w\|_{H^s(\Gamma_j)} \qquad \forall w \in H^s(\Gamma_j).$$

If $s > 0$, the norm on the right-hand side may be replaced by the seminorm $|w|_{H^s(\Gamma_j)}$. Also, the same estimate holds if the norm on the left-hand side is replaced by the "tilde" norm $\|w - w_h\|_{\widetilde{H}^\sigma(\Gamma_j)}$.

In order to prove Theorem 4.6, we use a *quasi-interpolation operator*.

Definition 4.7 (quasi-interpolant)**.** The operator $Q_h : L^2(\Gamma_j) \to S_h^0(\Gamma_j)$ is defined by

$$(Q_h\, v,\, v_h)_{L^2(\Gamma_j)} \;=\; (v,\, v_h)_{L^2(\Gamma_j)} \qquad \forall v_h \in S_h^0(\Gamma_j).$$

We may also replace Γ_j by Γ.

Apparently, for an element $\tau \in \mathcal{T}_h(\Gamma)$, we have that

$$(Q_h v)_{|\tau} \;=\; \frac{1}{|\tau|} \int_\tau v\, ds, \tag{4.8}$$

i.e., the value of $Q_h v$ on τ is the *mean value* of v over τ. This also shows that $Q_h v_h = v_h$ for all $v_h \in S^h(\Gamma_j)$ and so, Q_h is a *projection* onto $S_h(\Gamma_j)$.

Lemma 4.8. *For $u \in H^s(\Gamma_j)$ with $s \in [0, 1]$, we have the error estimates*

$$\|u - Q_h u\|^2_{L^2(\Gamma_j)} \;\leq\; C \sum_{\tau \in \mathcal{T}_h(\Gamma_j)} h_\tau^{2s} |u|^2_{H^s(\tau)}\,, \tag{4.9}$$

$$\|u - Q_h u\|_{L^2(\Gamma_j)} \;\leq\; C\, h^s\, |u|_{H^s(\Gamma_j)}\,, \tag{4.10}$$

where $h_\tau = |\tau|^{1/(d-1)}$ denotes the local mesh size of τ and $|u|_{H^0(\tau)} := \|u\|_{L^2(\tau)}$.

Proof. Beforehand, we show that (4.10) follows from (4.9):

$$\|u - Q_h u\|^2_{L^2(\Gamma_j)} \;\leq\; C \sum_{\tau \in \mathcal{T}_h(\Gamma_j)} \underbrace{h_\tau^{2s}}_{\leq h^{2s}} |u|^2_{H^s(\tau)} \;\leq\; C\, h^{2s} \sum_{\tau \in \mathcal{T}_h(\Gamma_j)} |u|^2_{H^s(\tau)}\,.$$

From the definition(s) of $|\cdot|_{H^s(\tau)}$, $s \in [0,1]$, we see that the sum on the right-hand side is at least bounded from above by $|u|^2_{H^s(\Gamma_j)}$.

Case $s = 0$. By the definition of Q_h, we obtain the Galerkin orthogonality

$$(u - Q_h u,\, v_h)_{L^2(\Gamma_j)} \;=\; 0 \qquad \forall v_h \in S_h^0(\Gamma_j), \tag{4.11}$$

showing that the projection Q_h is $L^2(\Gamma_j)$-*orthogonal*. Therefore, Q_h must have an operator norm ≤ 1, or equally,

$$\|Q_h u\|_{L^2(\Gamma_j)} \;\leq \|u\|_{L^2(\Gamma_j)},$$

which proves (4.9) for the case $s = 0$. Indeed,

$$\begin{aligned}\|u - Q_h u\|^2_{L^2(\Gamma_j)} \;&=\; (u - Q_h u,\, u - Q_h u)_{L^2(\Gamma_j)} \\ &\overset{(4.11)}{=} (u - Q_h u,\, u)_{L^2(\Gamma_j)} \overset{\text{C.S.}}{\leq} \|u - Q_h u\|_{L^2(\Gamma_j)}\, \|u\|_{L^2(\Gamma_j)}\,.\end{aligned}$$

Case $s \in (0,1)$. Let $\tau \in \mathcal{T}_h(\Gamma_j)$ and $x \in \tau$. From (4.8), we obtain

$$\begin{aligned}\big|u(x) - (Q_h u)(x)\big|^2 \;&=\; \Big|u(x) - \frac{1}{|\tau|}\int_\tau u(y)\,ds_y\Big|^2 \;=\; \frac{1}{|\tau|^2}\Big(\int_\tau u(x) - u(y)\,ds_y\Big)^2 \\ &=\; \frac{1}{|\tau|^2}\Big(\int_\tau \frac{u(x)-u(y)}{|x-y|^{(d-1)/2+s}}\,|x-y|^{(d-1)/2+s}\,ds_y\Big)^2 \\ &\overset{\text{C.S.}}{\leq} \frac{1}{|\tau|^2}\int_\tau \frac{|u(x)-u(y)|^2}{|x-y|^{d-1+2s}}\,ds_y \int_\tau \underbrace{|x-y|}_{\leq \operatorname{diam}(\tau)}{}^{(d-1+2s)}\,ds_y\,.\end{aligned}$$

Integrating the estimate over τ with respect to x, we obtain

$$\|u - Q_h u\|^2_{L^2(\tau)} \;\leq\; \underbrace{\frac{\operatorname{diam}(\tau)^{d-1+2s}}{|\tau|}}_{\leq C\, h_\tau^{2s}}\, |u|^2_{H^s(\tau)}\,,$$

where we have used the definition of $|\cdot|_{H^s(\tau)}$ after Sobolev-Slobodeckij (see Remark 2.23) and the fact that $\operatorname{diam}(\tau) \simeq h_\tau$, due to shape-regularity. This proves (4.9) for $s \in (0,1)$.

Case $s = 1$. The main idea here is to start as in the case above and to write the difference $u(x) - u(y)$ as a line integral over the derivative (along the line). This directional derivative can be bounded in terms of the gradient. As an additional difficulty, one has to transform the possibly curved element τ to its corresponding straight/flat element in the parameter domain. A rigorous proof can be found in [Steinbach, Theorem 10.2]. □

The above L^2-estimate can now be used to derive estimates in negative norms using duality and the projection property of Q_h.

Lemma 4.9. *For $u \in H^s(\Gamma)$ with $s \in [0,\, 1]$ and for $\sigma \in [-1,\, 0)$,*

$$\begin{aligned} \|u - Q_h u\|^2_{H^\sigma(\Gamma_j)} &\leq C\, h^{-2\sigma} \sum_{\tau \in \mathcal{T}_h(\Gamma_j)} h_\tau^{2s}\, |u|^2_{H^s(\tau)}\,, \\ \|u - Q_h u\|_{H^\sigma(\Gamma_j)} &\leq C\, h^{s-\sigma}\, |u|_{H^s(\Gamma_j)}\,. \end{aligned}$$

The same estimates hold if $\|u - Q_h u\|^2_{H^\sigma(\Gamma_j)}$ is replaced by $\|u - Q_h u\|^2_{\widetilde{H}^\sigma(\Gamma_j)}$.

Proof. Using the definition of the dual norm and the fact that $u - Q_h u \in L^2(\Gamma_j) \subset H^{-1/2}(\Gamma_j)$, we obtain

$$\begin{aligned} \|u - Q_h u\|_{H^\sigma(\Gamma_j)} &= \sup_{v \in \widetilde{H}^{-\sigma}(\Gamma_j)\setminus\{0\}} \frac{(u - Q_h u,\, v)_{L^2(\Gamma_j)}}{\|v\|_{H^{-\sigma}(\Gamma_j)}} \\ &\overset{(4.11)}{=} \sup_{v \in \widetilde{H}^{-\sigma}(\Gamma_j)\setminus\{0\}} \frac{(u - Q_h u,\, v - Q_h v)_{L^2(\Gamma_j)}}{\|v\|_{H^{-\sigma}(\Gamma_j)}} \\ &\overset{\text{C.S.}}{\leq} \|u - Q_h u\|_{L^2(\Gamma_j)} \sup_{v \in \widetilde{H}^{-\sigma}(\Gamma_j)\setminus\{0\}} \frac{\|v - Q_h v\|_{L^2(\Gamma_j)}}{\|v\|_{H^{-\sigma}(\Gamma_j)}}\,. \end{aligned}$$

We now apply Lemma 4.8 twice:

$$\begin{aligned} \|u - Q_h u\|^2_{L^2(\Gamma_j)} &\overset{(4.9)}{\leq} C \sum_{\tau \in \mathcal{T}_h(\Gamma_j)} h_\tau^{2s}\, |u|^2_{H^s(\tau)}\,, \\ \|v - Q_h v\|_{L^2(\Gamma_j)} &\overset{(4.10)}{\leq} C\, h^{-\sigma}\, |v|_{H^{-\sigma}(\Gamma_j)} \leq C\, h^{-\sigma}\, \|v\|_{\widetilde{H}^{-\sigma}(\Gamma_j)}\,. \end{aligned}$$

Substituting these estimates into the previous one yields the desired result. Estimates for the "tilde" norm $\|u - Q_h u\|_{\widetilde{H}^\sigma(\Gamma_J)}$ are obtained analogously. □

Lemma 4.8 and Lemma 4.9 together imply the statement of Theorem 4.6 for $-1 \leq \sigma \leq 0 \leq s \leq 1$. A proof of the remaining case $-1 \leq \sigma \leq s < 0$ requires another quasi-interpolation operator which is H^σ-orthogonal as well as techniques from interpolation theory, see, e.g., [Steinbach, Theorem 10.4].

A Priori Error Estimates for the Conormal

Recall the Céa estimate (4.7),

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \leq C \inf_{w_h \in S_h^0(\Gamma)} \|t - w_h\|_{H^{-1/2}(\Gamma)}\,.$$

Assumption: $t \in H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [-\frac{1}{2}, 1]$. By Exercise 26 (page 69), $t \in H^{-1/2}_{\mathrm{pw}}(\Gamma)$. The function w_h is piecewise constant w.r.t. $\mathcal{T}_h(\Gamma)$. Hence, it follows that each restriction $(t - w_h)_{|\Gamma_j}$ lies in $\widetilde{H}^{-1/2}(\Gamma_j)$, cf. Definition 4.4. Because w_h is piecewise constant, the infimum can be localized:

$$\begin{aligned} \|t - t_h\|_{H^{-1/2}(\Gamma)} &\overset{\text{Lem. 4.5+Def. 4.4}}{\leq} C \inf_{w_h \in S^1_h(\Gamma)} \sum_{j=1}^{J} \|(t - w_h)_{|\Gamma_j}\|_{\widetilde{H}^{-1/2}(\Gamma_j)} \\ &\leq C \sum_{j=1}^{J} \inf_{w_h \in S^0_h(\Gamma_j)} \|t_{|\Gamma_j} - w_h\|_{\widetilde{H}^{-1/2}(\Gamma_j)} \,. \end{aligned} \tag{4.12}$$

From Theorem 4.6, we obtain

$$\inf_{w_h \in S^0_h(\Gamma_j)} \|t_{|\Gamma_j} - w_h\|_{\widetilde{H}^{-1/2}(\Gamma_j)} \;\leq\; C\, h^{1/2+s} \begin{cases} \|t_{|\Gamma_j}\|_{\widetilde{H}^s(\Gamma_j)} & \text{for } s \in [-\frac{1}{2}, 0), \\ \|t_{|\Gamma_j}\|_{H^s(\Gamma_j)} & \text{for } s \in [0, 1]. \end{cases} \tag{4.13}$$

The combination of (4.12), (4.13), and Definition 4.4 yields the following important result.

Theorem 4.10 (error estimate of the conormal)**.** *Let t and t_h be the solutions of* (4.1) *resp.* (4.3)*, and assume that $t \in H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [-\frac{1}{2}, 1]$. Then*

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \;\leq\; C\, h^{1/2+s}\, \|t\|_{H^s_{\mathrm{pw}}(\Gamma)} \,.$$

In the optimal case $t \in H^1_{\mathrm{pw}}(\Gamma)$, we get

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \;=\; \mathcal{O}(h^{3/2}) \qquad \text{as } h \to 0.$$

For piecewise constant ansatz functions, this estimate cannot be improved.

Regularity Theory*

Question: when is $t \in H^s_{\mathrm{pw}}(\Gamma)$, for $s \in [-\frac{1}{2}, 1]$?
Partial answer:

Lemma 4.11. *Let Γ be piecewise smooth (cf. Sect. 4.1.1) and suppose that $u \in H^1(\Omega)$ with $-\Delta u = f$ in $\mathcal{D}^*(\Omega)$. Then, for $s \in (0, 1]$,*

$$u \in H^{3/2+s}(\Omega) \quad \Longrightarrow \quad t \in H^s_{\mathrm{pw}}(\Gamma).$$

Proof. From the assumption, we get $\nabla u \in H^{1/2+s}(\Omega)$. One can show that the trace operator γ_0 is continuous from $H^{1/2+s}(\Omega)$ to $H^s_{\mathrm{pw}}(\Gamma)$. Since the normal vector is piecewise smooth and $t = \gamma_0(\nabla u \cdot n)$, the assertion follows. □

The regularity of the solution u itself is a difficult field, see, e.g., [Grisvard, Dauge]. In order to get $u \in H^{3/2+s}(\Omega)$, one needs subtle assumptions on the domain and its boundary, and regularity of the Dirichlet data, such as $g_D \in H^{1+s}(\Gamma)$.

A Priori Error Estimates for the Reconstructed Solution

Assume a homogeneous right-hand side, i.e., $f^{\text{int}} = 0$, let t and t_h be the solutions of (4.1) resp. (4.3), and let $u \in H^1(\Omega)$ denote the corresponding solution in the interior. Thanks to the representation formula,

$$u \;=\; \widetilde{V}t - \widetilde{W}g_D\,. \tag{4.14}$$

Using this identity as a motivation, we define the *reconstruction* $\widetilde{u}_h \in H^1(\Omega)$ of the solution u by

$$\widetilde{u}_h \;:=\; \widetilde{V}t_h - \widetilde{W}g_D\,. \tag{4.15}$$

If $t \in L^\infty(\Gamma)$, we obtain from Lemma 3.14 that the error $u - \widetilde{u}_h$ is continuous in the interior of Ω and

$$u(x) - \widetilde{u}_h(x) \;=\; \int_\Gamma G(x,\,y)\,\big[t(y) - t_h(y)\big]\,ds \qquad \forall x \in \Omega.$$

Since $x \neq \Gamma$, the fundamental kernel fulfills $G(x,\,\cdot) \in C^\infty(\Gamma) \subset H^{-\sigma}(\Gamma)$ for any $\sigma \in \mathbb{R}$. Hence,

$$|u(x) - \widetilde{u}_h(x)| \;\leq\; \|G(x,\,\cdot)\|_{H^{-\sigma}(\Gamma)}\,\|t - t_h\|_{H^\sigma(\Gamma)}\,.$$

Choosing $\sigma = -\frac{1}{2}$, we see that the pointwise error converges at least as good as the error in the conormal.

Remark* 4.12. Better estimates are obtained via bounds of the error in a weaker norm, i.e., $\|t - t_h\|_{H^\sigma(\Gamma)}$ with $\sigma < -\frac{1}{2}$, using an Aubin-Nitsche argument. Details can be found in [Steinbach, Theorem 12.3]. In the optimal case $t \in H^1_{\text{pw}}(\Gamma)$, one finally obtains

$$|u(x) - \widetilde{u}_h(x)| \;=\; \mathcal{O}(h^3) \qquad \text{as } h \to 0, \qquad \text{for } x \in \Omega.$$

One can also analyze the H^1-error of the reconstruction.

Theorem 4.13. *Let u and $\widetilde{u}_h$ be defined according to* (4.14) *resp.* (4.15) *and assume that the bilinear form $a(\cdot,\cdot)$ is $H^1_0(\Omega)$-elliptic. Then there exists a constant C independent of h such that*

$$\|u - \widetilde{u}_h\|_{H^1(\Omega)} \;\leq\; C\,\|t - t_h\|_{H^{-1/2}(\Gamma)}\,.$$

Proof. Set $\widetilde{g}_h := \gamma_0 \widetilde{u}_h$ (interior trace). From the properties of the surface potentials and boundary integral operators, we obtain

$$\begin{aligned} g_D &= Vt + (\tfrac{1}{2}I + K)g_D\,, \\ \widetilde{g}_h &= Vt_h + (\tfrac{1}{2}I + K)g_D\,, \\ g_D - \widetilde{g}_h &= V(t - t_h). \end{aligned} \tag{4.16}$$

Thanks to the assumption on $a(\cdot,\cdot)$, the Dirichlet problem is well-posed, and so u and u_h can be characterized by

$$\begin{aligned} u = u_0 + \mathcal{E}g_D, \qquad & u_0 \in H^1_0(\Omega): \quad a(u_0 + \mathcal{E}g_D,\, v_0) = 0 \qquad \forall v_0 \in H^1_0(\Omega), \\ \widetilde{u}_h = \widetilde{u}_0 + \mathcal{E}\widetilde{g}_h, \qquad & \widetilde{u}_0 \in H^1_0(\Omega): \quad a(\widetilde{u}_0 + \mathcal{E}\widetilde{g}_h,\, v_0) = 0 \qquad \forall v_0 \in H^1_0(\Omega), \end{aligned} \tag{4.17}$$

where $\mathcal{E}$ is the bounded extension operator from the inverse trace theorem (Thm. 2.26). Subtraction yields

$$a(u_0 - \widetilde{u}_0,\, v_0) \;=\; a(\mathcal{E}(\widetilde{g}_h - g_D),\, v_0) \qquad \forall v_0 \in H^1_0(\Omega).$$

An application of Lax-Milgram provides the estimate

$$\|u_0 - \widetilde{u}_0\|_{H^1(\Omega)} \;\le\; \frac{\overline{c}_a}{\underline{c}_a}\, \|\mathcal{E}(\widetilde{g}_h - g_D)\|_{H^1(\Omega)}\,. \tag{4.18}$$

where $\overline{c}_a$, $\underline{c}_a$ are the constants of H^1-boundedness and H^1_0-ellipticity of $a(\cdot,\cdot)$, respectively. Finally, by the characterization (4.17) of u, $\widetilde{u}_h$, the triangle inequality, and estimate (4.18), we get

$$\begin{aligned} \|u - \widetilde{u}_h\|_{H^1(\Omega)} \;&\le\; \|u_0 - \widetilde{u}_0\|_{H^1(\Omega)} + \|\mathcal{E}(g_D - \widetilde{g}_h)\|_{H^1(\Omega)} \\ &\le\; \Big(\frac{\overline{c}_a}{\underline{c}_a} + 1\Big)\, \|\mathcal{E}(\widetilde{g}_h - g_D)\|_{H^1(\Omega)} \\ &\le\; \Big(\frac{\overline{c}_a}{\underline{c}_a} + 1\Big)\, C_{\mathrm{IT}}\, \|\underbrace{\widetilde{g}_h - g_D}_{=V(t-t_h)}\|_{H^{1/2}(\Gamma)} \;\le\; C\, \|t - t_h\|_{H^{-1/2}(\Gamma)}\,, \end{aligned}$$

where in the last line, we have used the inverse trace inequality, identity (4.16), and boundedness of the single layer potential operator V. □

Combining the last theorem with Theorem 4.10, we obtain the convergence result

$$\|u - \widetilde{u}_h\|_{H^1(\Omega)} \;=\; \mathcal{O}(h^{1/2+s}) \qquad \text{as } h \to 0,$$

under the assumption that $t \in H^s_{\mathrm{pw}}(\Gamma)$ for $s \in [-\frac{1}{2}, 1]$. In the optimal case, we get $\mathcal{O}(h^{3/2})$.

4.2.2 The Neumann Problem

For simplicity, we consider only the homogeneous Laplace equation. Analogous error estimates hold for $Lu = -\mathrm{div}(A\nabla u) + c\,u$ with $c \geq 0$. Recall Equations (4.2) and (4.4):

$$\begin{aligned}&\text{find } v \in H^{-1/2}(\Gamma): \quad \langle Dv,\, \varphi\rangle_\Gamma = \langle (\tfrac{1}{2}I - K')g_N,\, \varphi\rangle_\Gamma \quad \forall \varphi \in H^{1/2}(\Gamma),\\ &\text{find } v_h \in S^1_h(\Gamma): \quad \langle Dv_h,\, \varphi_h\rangle_\Gamma = \langle (\tfrac{1}{2}I - K')g_N,\, \varphi_h\rangle_\Gamma \quad \forall \varphi_h \in S^1_h(\Gamma).\end{aligned}$$

Recall also that $\ker(D) = \mathrm{span}(1_\Gamma)$ and that the Neumann data g_N must fulfill the compatibility condition $g_N \in H_*^{-1/2}(\Gamma)$.

Remark 4.14. In practice, one often uses regularizations of the operator D in order to remove the kernel, e.g.,

(a) $\langle \widetilde{D}v,\, \varphi\rangle_\Gamma := \langle Dv,\, \varphi\rangle_\Gamma + \langle w_{\mathrm{eq}},\, v\rangle_\Gamma \langle w_{\mathrm{eq}},\, \varphi\rangle_\Gamma,$

(b) $\langle \widetilde{D}v,\, \varphi\rangle_\Gamma := \langle Dv,\, \varphi\rangle_\Gamma + \int_\Gamma v\,ds \int_\Gamma \varphi\,ds.$

Exercise 27. Show that when replacing D by $\widetilde{D}$ in the above equations, v, v_h are still solutions of the original problems. For Choice (a), they are unique in $H_*^{1/2}(\Gamma)$. For Choice (b), they fulfill $\int_\Gamma v\,ds = \int_\Gamma v_h\,ds = 0$.

At least, D is elliptic on the factor space $H^{1/2}(\Gamma)_{/\ker(D)}$, and so we conclude by Céa's lemma that

$$|v - v_h|_{H^{1/2}(\Gamma)} \;\leq\; C \inf_{\varphi_h \in S^1_h(\Gamma)} |v - \varphi_h|_{H^{1/2}(\Gamma)}\,. \tag{4.19}$$

Theorem 4.15 (approximation of S^1_h, without proof)**.** *Let Ω be a $C^{k,1}$-domain. Then, for $s \in (\frac{1}{2}, 2]$ with $s \leq \max(\frac{3}{2}, k+1)$,*

$$\inf_{\varphi_h \in S^1_h(\Gamma)} \|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)} \;\leq\; C\,h^{s-1/2}\,|\varphi|_{H^s(\Gamma)} \qquad \forall \varphi \in H^s(\Gamma).$$

Combining (4.19) and Theorem 4.15, we get the following result.

Theorem 4.16 (error estimate for the Dirichlet trace)**.** *Let Ω be a $C^{k,1}$-domain and let v, v_h be solutions of* (4.2)*, resp.* (4.4)*. If the exact Dirichlet trace fulfills $v \in H^s(\Gamma)$ for some $s \in (\frac{1}{2}, 2]$ with $s \leq \max(\frac{3}{2}, k+1)$, then there exists a constant C independent of h such that*

$$|v - v_h|_{H^{1/2}(\Gamma)} \;\leq\; C\,h^{s-1/2}\,.$$

In the optimal case where $v \in H^2(\Gamma)$ (this is if the solution lies in $H^{5/2}(\Omega)$), the error is $\mathcal{O}(h^{3/2})$. This result cannot be improved, at least not with continuous piecewise linear ansatz functions.

For further estimate see, e.g., [Steinbach, Sect. 12.2].

4.3 Calculation of Matrix Entries

4.3.1 Single and Double Layer Potential

Opposed to Collocation BEM, the Galerkin BEM requires the evaluation of *double integrals*. In principle, there are three classes of evaluation formulae:

- analytic (only available for very simple situations)
- numerical quadrature (for both integrals)
- semi-analytic (analytic for the inner, quadrature for the outer integral)

Example for a Semi-Analytic Formula (Single Layer Potential)

We consider the single layer potential for the Laplace operator $L = -\Delta$ in two dimensions. Assume that

- the space $S_h^0(\Gamma)$ of piecewise constants is used for the approximation of the Neumann trace,
- each boundary element $\tau \in \mathcal{T}_h(\Gamma)$ is a straight line,
- the outer integral is approximated by the *midpoint rule* (for simplicity; other quadrature rules are treated similarly).

Let $i,\ j = 1, \ldots, n_h^{\mathrm{el}}$ be element indices, and let y_j denote the midpoint of element τ_j. Then,

$$[V_h]_{ji} \;=\; \int_{\tau_i}\int_{\tau_j} -\frac{1}{2\pi}\log|x-y|\,ds_x\,ds_y \overset{\text{hopefully}}{\approx} -\frac{|\tau_i|}{2\pi}\int_{\tau_j}\log|x-y_i|\,ds_x\,.$$

We remark that the integral above equals (up to the constant factor of $|\tau_j|$) that of a Collocation BEM with collocation points in the element midpoints.

Case 1: $i = j$. Only in this case, we have to evaluate an improper integral:

$$\begin{aligned}\int_{\tau_i}\log|x-y_i|\,ds_x \;&=\; \int_{-h_i/2}^{h_i/2}\log|\xi|\,d\xi \;=\; 2\lim_{\varepsilon\to 0}\int_{\varepsilon}^{h_i/2}\log|\xi|\,d\xi\\ &=\; 2\lim_{\varepsilon\to 0}\Big[\xi\,\log(\xi)-\xi\Big]_{\varepsilon}^{h_i/2}\\ &=\; h_i(\log(\tfrac{h_i}{2})-1) - \underbrace{\lim_{\varepsilon\to 0} 2(\varepsilon\log\varepsilon-\varepsilon)}_{=0}\,,\end{aligned}$$

where the last limit can be calculated with de l'Hospital's theorem.

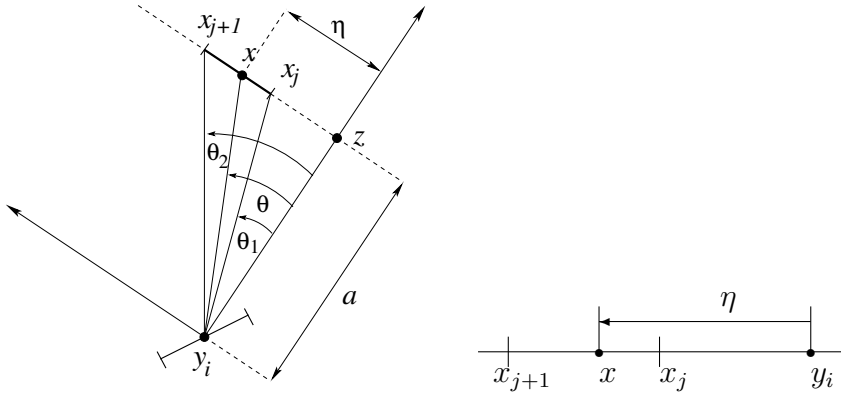


Figure 4.3: *Left:* Case 2 with coordinate transformation. *Right:* Case 3.

Case 2: $i \neq j$, y_i not collinear with τ_j. Let x_j, x_{j+1} denote the endpoints of τ_j and let us introduce a coordinate transformation as in Figure 4.3 (left):

$$\left.\begin{aligned} a &= |z - y_i| \\ \cos\theta &= \frac{|z - y_i|}{|x - y_i|} \end{aligned}\right\} \quad \Longrightarrow \quad |x - y_i| = \frac{a}{\cos\theta}$$

$$\eta = a\,\tan\theta$$

$$ds_x = d\eta \overset{(*)}{=} a\,\frac{d}{d\theta}\big(\tan\theta\big)\,d\theta = \frac{a}{\cos^2\theta}\,d\theta \overset{(**)}{=} a\,(1+\tan^2(\theta))\,d\theta$$

Now, the integral can be calculated with the usual transformation rules:

$$\int_{\tau_j} \log|x - y_i|\,ds_x \overset{(*)}{=} a\int_{\theta_1}^{\theta_2} \log\Big(\frac{a}{\cos\theta}\Big)\ \overset{\curvearrowleft}{\frac{d}{d\theta}}\ \big(\underbrace{\tan\theta}_{=\frac{\sin\theta}{\cos\theta}}\big)\,d\theta$$

$$= a\Big[\log\Big(\frac{a}{\cos\theta}\Big)\,\tan\theta\Big]_{\theta_1}^{\theta_2} - a\ \underbrace{\int_{\theta_1}^{\theta_2} \frac{\cos\theta}{a}\ \frac{(-a)\,(-\sin\theta)}{\cos^2\theta}\ \frac{\sin\theta}{\cos\theta}\,d\theta}_{= \int_{\theta_1}^{\theta_2}\tan^2(\theta)\,d\theta \overset{(**)}{=} \big[\tan\theta - \theta\big]_{\theta_1}^{\theta_2}}$$

$$= a\Big[\tan\theta\,\Big(\log\Big(\frac{a}{\cos\theta}\Big) - 1\Big) + \theta\Big]_{\theta_1}^{\theta_2}$$

Case 3: $i \neq j$, y_i collinear with τ_j. Assume, e.g., that y_i is on the side of x_j as shown in Figure 4.3 (right). Then,

$$\int_{\tau_j} \log|x - y_i|\, ds_x \;=\; \int_{|x_j - y_i|}^{|x_{j+1} - y_i|} \log|\eta|\, d\eta \;=\; \Big[\eta\,(\log\eta - 1)\Big]_{|x_j - y_i|}^{|x_{j+1} - y_i|}.$$

Numerical Quadrature

For continuous integrands, numerical quadrature is well-understood. However, the integral kernel occurring in V_h, K_h is singular. Thanks to Sauter, Schwab, and coworkers, there exists a systematic way (based on proper case distinction) to transform the double integrals appearing in V_h and K_h to integrals over $[0,\,1]^{2(d-1)}$ with an *analytic (and therefore continuous) integrand*. The latter can be treated with conventional quadrature rules. Details can be found in [Sauter/Schwab, Sect. 5.2] or [Erichsen/Sauter].

4.3.2 Representation of the Hypersingular Operator D

In this subsection, we treat the Laplace operator $L = -\Delta$ and assume again that Γ is piecewise smooth with parts $\Gamma_j \in C^1$ (cf. Sect. 4.1.1).

Definition 4.17 (2D surface curl)**.** Let $d = 2$. For $\widetilde{v} \in C^1(\overline{\Omega})$, we define

$$\operatorname{curl}\widetilde{v} \;:=\; \begin{bmatrix} \partial_2\widetilde{v} \\ -\partial_1\widetilde{v} \end{bmatrix},$$

where ∂_k is a short hand for $\partial/\partial x_k$. For $v \in C^1(\Gamma_j)$, we define

$$(\operatorname{curl}_{\Gamma_j} v)(x) \;:=\; n(x) \cdot \operatorname{curl}\widetilde{v}(x), \qquad \text{for } x \in \Gamma_j\,,$$

where $\widetilde{v} \in C^1(\overline{\Omega})$ is an extension of v from Γ_j to $\overline{\Omega}$.

Definition 4.18 (3D surface curl)**.** Let $d = 3$. For $v \in C^1(\Gamma_j)$ we define

$$(\operatorname{curl}_{\Gamma_j} v)(x) \;:=\; n(x) \times \nabla\widetilde{v}(x), \qquad \text{for } x \in \Gamma_j\,,$$

where $\widetilde{v} \in C^1(\overline{\Omega})$ is an extension of v.
Note that $-n \times (n \times f) = f - (f \cdot n)n$ for vector fields f. So $(\operatorname{curl}_{\Gamma_j} v)(x)$ is (up to rotation) the projection of $\nabla\widetilde{v}(x)$ to the tangent plane defined by $n(x)$.

Lemma 4.19. *The above definitions are independent of the choice of the extension.*

Proof for $d = 2$.* Let $\Gamma_j = \{y(t) \in \mathbb{R}^2 : t \in (0,\, 1)\}$. Then

$$ds_y = |y'(t)|\,dt, \qquad n(y(t)) = \frac{1}{|y'(t)|}\begin{bmatrix} y_2'(t) \\ -y_1'(t) \end{bmatrix} \qquad \text{(if positive orientation)}.$$

Hence, for $\varphi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\Gamma_j} \operatorname{curl}_{\Gamma_j}\varphi\, ds_y &= \int_0^1 \frac{1}{\cancel{|y'(t)|}}\begin{bmatrix} y_2'(t) \\ -y_1'(t) \end{bmatrix} \cdot (\operatorname{curl}\widetilde{v})(y(t))\,\cancel{|y'(t)|}\,\varphi(y(t)\,dt \\ &= \int_0^1 (\nabla\widetilde{v})(y(t)) \cdot y'(t)\,\varphi(y(t)\,dt = \int_0^1 \frac{d}{dt} v(y(t))\,\varphi(y(t)\,dt.\end{aligned}$$

We see that $\operatorname{curl}_{\Gamma_j}$ is essentially the tangential derivative. □

For $v \in C^1_{\mathrm{pw}}(\Gamma)$ (with respect to the partition $\Gamma = \bigcup_{j=1}^J \overline{\Gamma}_j$), we define $\operatorname{curl}_\Gamma v \in L^2(\Gamma)$ by

$$(\operatorname{curl}_\Gamma v)_{|\Gamma_j} = \operatorname{curl}_{\Gamma_j} v, \qquad \text{for } j = 1, \ldots, J.$$

Theorem 4.20 (without proof)**.** *Suppose that $L = -\Delta$, $\Gamma \in C^1_{\mathrm{pw}}$ and u, $v \in H^{1/2}(\Gamma) \cap C(\Gamma) \cap C^1_{\mathrm{pw}}(\Gamma)$. Then, for $d = 2$,*

$$\langle D\,u,\, v\rangle_\Gamma = -\frac{1}{2\pi}\int_\Gamma \operatorname{curl}_\Gamma v(x) \int_\Gamma \log|x-y|\,\operatorname{curl}_\Gamma u(y)\,ds_y\,ds_x\,.$$

For $d = 3$,

$$\langle D\,u,\, v\rangle_\Gamma = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{\operatorname{curl}_\Gamma v(x) \cdot \operatorname{curl}_\Gamma u(y)}{|x-y|}\,ds_y\,ds_x\,.$$

Remark 4.21.

1. In short, we may write
$$\langle D\,u,\, v\rangle_\Gamma = \langle \operatorname{curl}_\Gamma v,\, V \operatorname{curl}_\Gamma u\rangle_\Gamma\,.$$
2. Recall that $S^1_h(\Gamma) \subset H^{1/2}(\Gamma) \cap C(\Gamma) \cap C^1_{\mathrm{pw}}(\Gamma)$. Hence, the entries of the hypersingular matrix D_h can be reduced to those of the single layer potential matrix V_h.
3. A proof can be found in [Steinbach, Sect. 6.5] and is based on the *integration by parts formula*
$$\int_\Gamma v(y)\,\operatorname{curl}_\Gamma w(y)\,ds_y = -\int_\Gamma \operatorname{curl}_\Gamma v(y)\,w(y)\,ds_y + \sum_{j=1}^J v(y_j(t))\,w(y_j(t))\Big|_0^1,$$
if $\Gamma_j = \{y_j(t) \in \mathbb{R}^2 : t \in (0,\, 1)\}$ (with positive orientation). We see that if v, $w \in C(\Gamma)$, then the sum above can be dropped. This is the background of phrases like "we use integration by parts for the hypersingular integral operator". A similar formula holds for the Helmholtz operator, cf. [McLean, Thm. 9.15].

4.4 Complexity of Galerkin BEM

As pointed out earliear, the BEM matrices V_h, K_h, D_h are *dense*. Hence, *assembling* of these matrices requires $\mathcal{O}(n_h^2)$ floating point operations, where n_h is the number of nodes/elements of the triangulation. E.g., if the solution of the system

$$V_h \underline{t}_h \;=\; (\tfrac{1}{2}M_h - K_h)\underline{g}_{D,h}$$

is obtained by Gauss' algorithm, the computational work is proportional to

$$\mathcal{O}(n_h^3) \;=\; \mathcal{O}(h^{-3(d-1)}),$$

where here and below, we assume that $\Omega \simeq (0,\,1)^d$ and that $\mathcal{T}_h(\Gamma)$ is quasi-uniform triangulation.

For a quasi-uniform triangulation of Ω, a finite element method leads to $n_{h,\mathrm{FEM}} = \mathcal{O}(h^{-d})$ many unknowns. Factorization by a *sparse* direct solver (pivoting, nested dissection, etc., such as PARDISO) requires $\mathcal{O}(n_{h,\mathrm{FEM}}^{3/2})$ operations if $d = 2$, and $\mathcal{O}(n_{h,\mathrm{FEM}}^2)$ if $d = 3$. If an optimal solver (e.g., iterative solver with multilevel/multigrid preconditioner) the complexity (of FEM) changes to $\mathcal{O}(n_{n,\mathrm{FEM}})$. We compare:

		direct	iterative	
2D	unknowns	memory/operations	memory	operations
FEM	h^{-2}	h^{-3} (sparse)	h^{-2}	h^{-2}
BEM	h^{-1}	h^{-3} (Gauss)	h^{-2}	?

		direct	iterative	
3D	unknowns	memory/operations	memory	operations
FEM	h^{-3}	h^{-6}	h^{-3}	h^{-3}
BEM	h^{-2}	h^{-6}	h^{-4}	?

This comparison is probably unfair in many ways: (1) for FEM, extremely good solvers have been considered, (2) in BEM we haven't included any reconstruction of the solution. We see, however, that in order to outperform the FEM, in BEM, one needs

- a cheaper way to *store and factorize or apply* a BEM matrix, e.g., $\mathcal{O}(n_h \log^\alpha(n_h))$,
- a good preconditioner with comparable complexity.

For such techniques, see Chapter 5. In view of iterative solvers and preconditioning, the conditioning of V_h and D_h is of interest.

4.4.1 Conditioning of V_h

Theorem 4.22. *Assume that V is elliptic, that $\mathcal{T}_h(\Gamma)$ is quasi-uniform, and let V_h denote the matrix corresponding to V with respect to the canonical basis of $S_h^0(\Gamma)$. Then there exist constants $c_1,\ c_2 > 0$ independent of h such that*

$$c_1\, h^d\, \|\underline{w}_h\|_{\ell^2}^2 \ \leq\ (V_h\, \underline{w}_h,\, \underline{w}_h)_{\ell^2} \ \leq\ c_2\, h^{d-1}\, \|\underline{w}_h\|_{\ell^2}^2 \qquad \forall \underline{w}_h \in \mathbb{R}^{n_h^{\mathrm{el}}}.$$

Hence,

$$\kappa(V_h) \ =\ \mathcal{O}(h^{-1}).$$

For comparison: for a quasi-uniform triangulation of Ω, the condition number of a FEM stiffness matrix is $\mathcal{O}(h^{-2})$.

Proof.* Let $w_h \in S_h^0(\Gamma)$ correspond to the vector $\underline{w}_h \in \mathbb{R}^{n_h^{\mathrm{el}}}$. Then

$$(V_h\, \underline{w}_h,\, \underline{w}_h)_{\ell^2} \ =\ \langle w_h,\, V w_h\rangle_\Gamma\,.$$

Upper bound. Since $w_h \in L^2(\Gamma) \subset H^{-1/2}(\Gamma)$,

$$\|w_h\|_{H^{-1/2}(\Gamma)} = \sup_{v\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{(w_h,\, v)_{L^2(\Gamma)}}{\|v\|_{H^{1/2}(\Gamma)}} \leq \sup_{v\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{(w_h,\, v)_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}} \overset{\text{C.S.}}{\leq} \|w_h\|_{L^2(\Gamma)}\,.$$

Since V is bounded in $H^{-1/2}(\Gamma)$,

$$\begin{aligned}\langle w_h,\, V w_h\rangle_\Gamma \ &\leq\ C\,\|w_h\|_{H^{-1/2}(\Gamma)}^2 \ \leq\ C\,\|w_h\|_{L^2(\Gamma)}^2 \\ &\leq\ C\sum_{i=1}^{n_h^{\mathrm{el}}} w_i^2\, \underbrace{|\tau_i|}_{\lesssim h_i^{d-1}} \ \leq\ C\,h^{d-1}\,\|\underline{w}_h\|_{\ell^2}^2\,,\end{aligned}$$

where $w_i = w_{h|\tau_i}$ are the entries of $\underline{w}_h$.

Lower bound. We need some tools (for details, see [Steinbach, Lemma 12.6]).

(i) We define the *space of local bubble functions*

$$S_h^B(\Gamma) \ :=\ \operatorname{span}(\varphi_i^B)_{i=1}^{n_h^{\mathrm{el}}} \ \subset H^{1/2}(\Gamma),$$

where the basis functions φ_i^B are defined via the reference interval $(0,\,1)$ and triangle $\{(x,\,y)\in[0,\,1]^2 : y \leq 1-x\}$ by

$$\varphi^B(\xi) \ :=\ \begin{cases}\xi(1-\xi) & \text{if } d=2,\\ \xi_1\xi_2(1-\xi_1-\xi_2) & \text{if } d=3.\end{cases}$$

(ii) We have the *inverse inequality*

$$\|v_h\|_{H^{1/2}(\Gamma)} \;\leq\; C\,h^{-1/2}\,\|v_h\|_{L^2(\Gamma)} \qquad \forall v_h \in S_h^B(\Gamma),$$

which can, e.g., be obtained from an interpolation of corresponding estimates in the L^2- and H^1-norm.

(iii) We define $Q_h^B : L^2(\Gamma) \to S_h^B(\Gamma)$ by the relation

$$\int_{\tau_i} (Q_h^B w)\,ds \;=\; \int_{\tau_i} w\,ds \qquad \forall i = 1,\dots,n_h^{\mathrm{el}}\,, \qquad \forall w \in L^2(\Gamma).$$

It follows immediately that for all (piecewise constant) $w_h \in S_h^0(\Gamma)$,

$$(w_h,\, Q_h^B w_h)_{L^2(\Gamma)} \;=\; \sum_{i=1}^{n_h^{\mathrm{el}}} w_{|\tau_i} \underbrace{\int_{\tau_i} Q_h^B w_h\,ds}_{=|\tau_i|\,w_{|\tau_i}} \;=\; \|w_h\|_{L^2(\Gamma)}^2\,.$$

(iv) The operator Q_h^B fulfills the stability estimate

$$\|Q_h^B w\|_{L^2(\Gamma)} \;\leq\; \sqrt{2}\,\|w\|_{L^2(\Gamma)} \qquad \forall w \in L^2(\Gamma).$$

Now,

$$\begin{aligned}
\|w_h\|_{H^{-1/2}(\Gamma)} \;&=\; \sup_{v\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{\langle w_h,\, v\rangle_\Gamma}{\|v\|_{H^{1/2}(\Gamma)}} \;\geq\; \frac{(w_h,\, Q_h^B w_h)_{L^2(\Gamma)}}{\|Q_h^B w_h\|_{H^{1/2}(\Gamma)}} \\
&\overset{\text{(iii)}}{=} \frac{\|w_h\|_{L^2(\Gamma)}^2}{\|Q_h^B w_h\|_{H^{1/2}(\Gamma)}} \;\overset{\text{(ii)}}{\geq}\; \frac{1}{C\,h^{-1/2}}\,\frac{\|w_h\|_{L^2(\Gamma)}^2}{\|Q_h w_h\|_{L^2(\Gamma)}} \;\overset{\text{(iv)}}{\geq}\; \frac{h^{1/2}}{C\sqrt{2}}\|w_h\|_{L^2(\Gamma)}\,.
\end{aligned}$$

Using the ellipticity of V, the above estimate, and quasi-uniformity of $\mathcal{T}_h(\Gamma)$, we obtain the desired lower bound

$$\begin{aligned}
\langle w_h,\, V w_h\rangle_\Gamma \;&\geq\; c_V\,\|w_h\|_{H^{-1/2}(\Gamma)}^2 \;\geq\; \frac{c_V}{2C^2}\,h\,\|w_h\|_{L^2(\Gamma)}^2 \\
&=\; \frac{c_V}{2C^2}\,h \sum_{i=1}^{n_h^{\mathrm{el}}} w_i^2 \underbrace{|\tau_i|}_{\simeq h^{d-1}} \;\simeq\; h^d\,\|\underline{w}_h\|_{\ell^2}^2\,.
\end{aligned}$$

This concludes the proof. □

4.4.2 Conditioning of D_h

Theorem 4.23. *Assume that $\mathcal{T}_h(\Gamma)$ is quasi-uniform, let D_h be denote the matrix corresponding to D with respect to the canonical basis of $S^1_h(\Gamma)$. Then for $Lu = -\mathrm{div}(A\nabla u) + c\,u$ with $c > 0$, there exist constants c_1, $c_2 > 0$ independent of h such that*

$$c_1\, h^{d-1}\, \|\underline{v}_h\|^2_{\ell^2} \;\leq\; (D_h\, \underline{v}_h,\, \underline{v}_h)_{\ell^2} \;\leq\; c_2 h^{d-2}\, \|\underline{v}_h\|^2_{\ell^2} \qquad \forall \underline{v}_h \in \mathbb{R}^{\mathrm{vert}}_h\,.$$

For $c = 0$, the same estimate holds when D_h is replaced by a suitable regularization $\widetilde{D}_h$, corresponding to Remark 4.14.

Proof.* Let $v_h \in S^1_h(\Gamma)$ correspond to the vector $\underline{v}_h$. Then

$$(D_h \underline{v}_h,\, \underline{v}_h)_{\ell^2} \;=\; \langle Dv_h,\, v_h\rangle_\Gamma\,.$$

Upper bound. Using that D is bounded in $H^{1/2}(\Gamma)$ and using the *inverse inequality*

$$\|v_h\|_{H^{1/2}(\Gamma)} \;\leq\; C\, h^{-1/2}\, \|v_h\|_{L^2(\Gamma)}\,,$$

we obtain

$$\langle Dv_h,\, v_h\rangle_\Gamma \;\lesssim\; \|v_h\|^2_{H^{1/2}(\Gamma)} \;\lesssim\; h^{-1}\, \|v_h\|^2_{L^2(\Gamma)} \;=\; h^{-1} \sum_{i=1}^{n_{\mathrm{el}}} \|v_h\|^2_{L^2(\tau_i)}\,.$$

Let $\mathcal{V}_i$ the set of vertices of τ_i (≤ 3 many). Due to shape-regularity, each vertex belongs to a uniformly bounded number of elements. Therefore,

$$\sum_{i=1}^{n_{\mathrm{el}}} \|v_h\|^2_{L^2(\tau_i)} \;\simeq\; h^{d-1} \sum_{k\in\mathcal{V}_i} v_i^2 \;\lesssim\; C\, h^{d-1}\, \|v_h\|^2_{\ell^2}\,,$$

which altogether implies the upper bound.

Lower bound. Since D is elliptic, similar arguments yield

$$\langle Dv_h,\, v_h\rangle_\Gamma \;\gtrsim\; \underbrace{\|v\|^2_{H^{1/2}(\Gamma)}}_{\geq \|v\|^2_{L^2(\Gamma)}} \;\gtrsim\; \sum_{i=1}^{n_h^{\mathrm{el}}} |\tau_i| v_i^2 \;\gtrsim\; h^{d-1}\, \|\underline{v}_h\|^2_{\ell^2}\,,$$

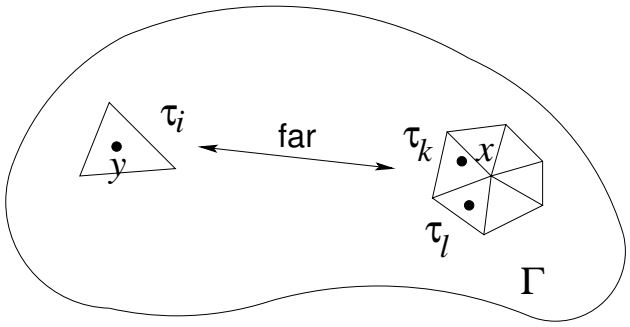
where quasi-uniformity was used. □

Chapter 5

Fast BEM*

5.1 Motivation

We have seen that the crucial problem is the *storage* of the matrices V_h, K_h, etc. For a quasi-uniform mesh we need $\mathcal{O}(h^{2(d-1)})$ memory where we have only $\mathcal{O}(h^{d-1})$ unknowns. The basic idea of fast boundary element methods is to *approximate* the matrices with less storage amount. Here we give a motivation why this can be possible. Let Γ be the boundary of a three-dimensional domain, let τ_i, τ_k, τ_ℓ, etc. be elements, where τ_i is *far* away from τ_k and all the remaining elements touch τ_k, see the figure below.



For $d = 3$, $L = -\Delta$, and the definition of V_h as in Sect. 4.1.4 we have

$$[V_h]_{ik} \;=\; \int_{\tau_i}\int_{\tau_k} \frac{1}{4\pi}\,\frac{1}{|x-y|}\,ds_x\,ds_y\,.$$

Since $1/|x-y|$ decays very fast if $|x-y|$ becomes large, we can write $|x-y| \approx |x_i^* - x_k^*|$ where x_i^* and x_k^* are the centers of τ_i and τ_k, respectively. Thus the double integral is approximated by a constant. For an element τ_ℓ touching τ_k, we have

$$[V_h]_{i\ell} \;\approx\; [V_h]_{ik}$$

because the difference $|x_i^* - x_k^*| \approx |x_i^* - x_\ell^*|$. In the figure above we can approximate *six* double integrals (six entries of the matrix) by *one* constant, which leads to a reduced storage amount. Using some tricks one can generalize this concept.

Among many fast BEM approaches are

- *wavelets* (here one constructs a special basis which leads to a sparse representation),
- the *fast multipole method* (using taylor expansion one can realize at least a fast application of the matrices),
- hierarchical matrices and data-sparse approximation

For more details see e. g., [Steinbach] and [Bebendorf]. We will follow the last approach. There the main tricks are

- low-rank matrices, and
- hierarchical clustering.

The hierarchical matrices were introduced by Hackbusch and Khoromskij. The data-sparse approximation that we will discuss is named adaptive cross approximation (ACA) and was introduced by Bebendorf and Rjasanow. Other techniques and references can be found in [Bebendorf], [Steinbach], and also online at `www.hlib.org`.

5.2 Low-rank Matrices

For a matrix $A \in \mathbb{R}^{m\times n}$ we define

$$\operatorname{range}(A) := \{A\,y : y \in \mathbb{R}^n\}\,, \qquad \operatorname{rank}(A) := \dim(\operatorname{range}(A))\,.$$

Lemma 5.1. *(i)* $\operatorname{rank}(A) \le \min(m,\, n) \quad \forall A \in \mathbb{R}^{m\times n}$

(ii) $\operatorname{rank}(A\,B) \le \min(\operatorname{rank}(A),\, \operatorname{rank}(B)) \quad \forall A \in \mathbb{R}^{m\times p}\,,\ B \in \mathbb{R}^{p\times n}$

(iii) $\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B) \quad \forall A,\, B \in \mathbb{R}^{m\times n}$

Definition 5.2. We define the *set of matrices with rank at most* k (in short: rank-k matrices),

$$\mathbb{R}_k^{m\times n} := \{A \in \mathbb{R}^{m\times n} : \operatorname{rank}(A) \le k\}\,.$$

Attention: $\mathbb{R}_k^{m\times n}$ does *not* form a linear space because addition of two rank-k matrices can increase the rank beyond k.

Lemma 5.3.

$$A \in \mathbb{R}_k^{m\times n} \quad \Longleftrightarrow \quad \exists U \in \mathbb{R}^{m\times k}\,,\ V \in \mathbb{R}^{n\times k} : A = U\,V^\top$$

The above representation $U\,V^\top$ of a rank-k matrix is called *outer product form*. Note that if we have such a representation, the matrix-vector multiplication

$$A\,y \;=\; U\,\underbrace{(V^\top y)}_{\in\mathbb{R}^k}$$

can be computed in $\mathcal{O}(k(m+n))$ operations. Also, the storage amount of U and V is only $k(m+n)$.

Definition 5.4. A matrix $A \in \mathbb{R}_k^{m\times n}$ is said to have *low rank* if

$$k(m+n) \;<\; m\,n\,.$$

Obviously we should represent (in particular store) low-rank matrices in outer product form.

Remark 5.5. Two low-rank matrices can be multiplied and added in low complexity. Also the singular value decomposition (SVD)

$$A \;=\; U\,\Sigma\,V^\top$$

(with $U \in \mathbb{R}^{m\times k}$, $V \in \mathbb{R}^{n\times k}$ orthogonal and $\Sigma \in \mathbb{R}^{k\times k}$ diagonal) of a low-rank matrix can be computed cheaply. Using that one, for A, $B \in \mathbb{R}_k^{m\times n}$, the best approximation C of the sum $A+B$ with respect to the Frobenious norm, i. e,

$$\|A+B-C\|_F \to \min_{C\in\mathbb{R}_k^{m\times n}}\,,$$

can be computed also efficiently. We refer to this approximated addition by *rounded addition*. It is a similar concept to the rounded addition of fixed floating point operations in processors, but here we do not cut the precision of a number but the rank of a matrix. The computational complexity of the rounded addition is $\mathcal{O}(k^2(m+n))$.

5.3 Degenerate Kernels

Unfortunately, boundary element matrices *cannot* be approximated by low-rank matrices (but as will turn out certain matrix blocks can). They would, if the kernel of the underlying integral operator had a different form.

Definition 5.6. Let D_1, D_2 be subsets of $\mathbb{R}^d$ or a manifold Γ. An integral kernel $\kappa : D_1 \times D_2 \to \mathbb{R}$ is called *degenerate* if there exists a constant $k \in \mathbb{N}$ and functions $u_\ell : D_1 \to \mathbb{R}$, $v_\ell : D_2 \to \mathbb{R}$ for $\ell = 1, \ldots, k$ such that

$$\kappa(x, y) \;=\; \sum_{\ell=1}^{k} u_\ell(x)\, v_\ell(y) \qquad \forall x \in D_1,\, y \in D_2\,.$$

The number k is called *degree of degeneracy*.

Suppose we have a matrix A defined by

$$A_{ij} \;=\; \int_\Gamma \int_\Gamma \kappa(x, y)\, \psi_j(y)\, \varphi_i(x)\, ds_x\, ds_y$$

and suppose we have index sets I and J and submanifolds D_1, $D_2 \subset \Gamma$ with

$$\operatorname{supp}(\varphi_i) \subset D_1 \quad \forall i \in I\,, \qquad \operatorname{supp}(\psi_j) \subset D_2 \quad \forall j \in J\,.$$

If κ (restricted to $D_1 \times D_2$) is degenerate of degree k then

$$A_{ij} = \sum_{\ell=1}^{k} \int_{D_1} u_\ell(x)\varphi_i(x)\, ds_x \int_{D_1} v_\ell(y)\psi_j(y)\, ds_y \qquad \forall i \in I,\, j \in J\,,$$

and so the block $[A_{ij}]_{i\in I,\, j\in J}$ has rank k.

The kernels appearing in our boundary integral operators are not degenerate. However, we can try to *approximate* them by degenerate kernels.

5.4 Asymptotically Smooth Kernels

Definition 5.7. An integral kernel $\kappa : D_1 \times \mathbb{R}^d \to \mathbb{R}$ with $\kappa(x, \cdot) \in C^\infty(\mathbb{R}^d \setminus \{x\})$ for all $x \in D_1$ is called *asymptotically smooth* in D_1 with respect to y if there exist constants γ, $c > 0$ such that for all $x \in D_1$ we have

$$\left|\partial_y^\alpha \kappa(x, y)\right| \;\le\; c\,|\alpha|!\,\gamma^{|\alpha|}\,\frac{|\kappa(x, y)|}{|x - y|^{|\alpha|}} \qquad \forall\, \text{multi-indices } \alpha \;\; \forall y \in \mathbb{R}^d \setminus \{x\}\,.$$

Remark 5.8. The kernels $U^*(x,\, y)$ and $\frac{\partial}{\partial n_x}U^*(x,\, y)$ appearing in V and K can be shown to be asymptotically smooth.

We want to exploit this smoothness for a Taylor expansion. However, in the vicinty of the singulary we cannot expect good convergence. Therefore we will assume that x and y are sufficiently far away from each other.

Definition 5.9. For sets D_1, D_2 and $x \in D_1$ we define the distances

$$\operatorname{dist}(x,\, D_2) := \inf_{y\in D_2} |x-y|\,, \qquad \operatorname{dist}(D_1,\, D_2) := \inf_{x\in D_1,\, y\in D_2} |x-y|\,.$$

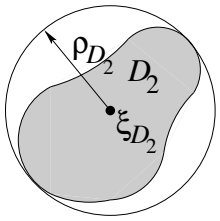
Assume now that a kernel $\kappa : D_1 \times D_2 \to \mathbb{R}$ is analytic with respect to the second argument (y) and assume at least that $\operatorname{dist}(D_1,\, D_2) > 0$. Then by Taylor's expansion we have

$$\kappa(x,\, y) \;=\; \underbrace{\sum_{|\alpha|<p} \frac{1}{\alpha!}\partial_y^\alpha \kappa(x,\, \xi_{D_2})\,(y-\xi_{D_2})^\alpha}_{=:T_p[\kappa](x,\,y)} + R_p(x,\, y)\,,$$

where

$$R_p(x,\, y) \;=\; \sum_{|\alpha|\geq p} \frac{1}{\alpha!}\partial_y^\alpha \kappa(x,\, \xi_{D_2})\,(y-\xi_{D_2})^\alpha$$

and ξ_{D_2} is the *Chebyshev center* of D_2, defined as the center of the ball of minimum radius that contains D_2. We denote the minimum radius by ρ_{D_2}, see below.



The next lemma clarifies how much error we make when cutting the Taylor series.

Lemma 5.10. *Let $\kappa : D_1 \times D_2 \to \mathbb{R}$ an integral kernel which is analytic with respect to y and let $\kappa : D_1 \times \mathbb{R}^d \to \mathbb{R}$ be asymptotically smooth. Furthermore, suppose that the condition*

$$\eta \operatorname{dist}(\xi_{D_1},\, D_2) \;\geq\; \rho_{D_2}$$

holds for some $\eta > 0$ with $2\,\gamma\,\sqrt{d}\,\eta < 1$. Then

$$\big|\kappa(x,\, y) - T_p[\kappa](x,\, y)\big| \;\leq\; \frac{(2\,\gamma\,\sqrt{d}\,\eta)^p}{1-2\,\gamma\,\sqrt{d}\,\eta}\,|\kappa(x,\, \xi_{D_2})|\,.$$

The lemma states that if D_1 is sufficiently far away from D_2, we have *exponential convergence* of the Taylor series as $p \to \infty$. I.e. there is hope that we can cut the series for a rather small p (which leads to a low rank representation) and to make only a small error.

Usually, we consider integral kernels which are analytic and asymptotically smooth with respect to both variables x and y. Then we use the symmetric condition

$$\min(\rho_{D_1},\, \rho_{D_2}) \;\leq\; \eta \operatorname{dist}(D_1,\, D_2) \tag{5.1}$$

instead of the condition in Lemma 5.10.

5.5 Admissible Blocks

For index sets $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$ we denote by $t \subset I$, $s \subset J$ *(index) blocks*. Note that the entries need not be contiguous (a block can also look like $\{2, 5, 7\}$). For $A \in \mathbb{R}^{n\times m} = \mathbb{R}^{I\times J}$ we define

$$A_{t\times s} := [A_{ij}]_{i\in t,\, k\in s}\,,$$

i. e., the restriction to the block $t \times s$. Finally, we denote by $|t|$ the number of indices in the block t.

Definition 5.11. Let $A_{t\times s}$ correspond to

$$\int_{D_1}\int_{D_2} \kappa(x,\, y)\, \psi_j(y)\, \varphi_i(x)\, ds_x\, ds_y \qquad \text{for } i \in t,\, j \in s\,,$$

with $\bigcup_{i\in t} \operatorname{supp}\varphi_i \subset D_1$ and $\bigcup_{j\in s} \operatorname{supp}\psi_j \subset D_2$. Then we call the block $t \times s$ *admissible* if (5.1) is fulfilled.

Lemma 5.12. *For an admissible block $t \times s$ and $\kappa : D_1 \times D_2 \to \mathbb{R}$ analytic and asymptotically smooth in both variables, the matrix block $A_{t\times s}$ (defined as above) can be approximated by a low-rank matrix $\widetilde{A}_{t\times s} \in \mathbb{R}_k^{t\times s}$. Let ε denote the approximation error (with respect to $\|\cdot\|_F$), then*

$$k \leq p^d \simeq |\log \varepsilon|^d\,,$$

where p denotes the order of the Taylor expansion.

The ultimate goal is to find a partition of $I \times J$ into blocks where we can use the Taylor expansion and the low-rank approximation.

Definition 5.13. A set $P \subset \mathcal{P}(I \times J)$ (where $\mathcal{P}(\cdot)$ denotes the power set) is called *partition* of $I \times J$ if

$$I \times J = \bigcup_{b \in P} b\,, \qquad \text{and} \qquad b_1 \neq b_2 \in P \implies b_1 \cap b_2 = \emptyset\,.$$

A block $b = t \times s$ of a partition is called *small* if $\min(|t|, |s|) < n_0$ where n_0 is a positive parameter. Finally, we call a partition P *admissible* if each block $b \in P$ is either admissible or small.

In the next two sections we construct admissible partitions of $I \times J$. There we restrict ourselves to a special class of partitions which are constructed by organizing the index sets I and J separately in a hierarchical tree structure.

5.6 Cluster Trees

Definition 5.14. (i) Let $T = (\mathcal{V}, \mathcal{E})$ be a *tree* (i. e., a simple, directed, and connected graph with no cycles). Here $\mathcal{V}$ and $\mathcal{E}$ are the set of vertices and edges of the graph, respectively. For a vertex $t \in \mathcal{V}$ we define the *set of sons*

$$\mathcal{S}(t) := \{t' \in \mathcal{V} : (t, t') \in \mathcal{E}\}\,.$$

The *root* of the tree is a unique vertex which is not a son of any other vertex. We call a vertex a *leaf* if it has no sons. The set of leaves is

$$\mathcal{L}(T) := \{t \in \mathcal{V} : \mathcal{S}(t) = \emptyset\}\,.$$

We define the *level* of a vertex t: if t is the root, $\operatorname{level}(t) := 0$, for all other vertices t, $\operatorname{level}(t)$ is the minimal number of edges connecting t and the root. Finally, the *depth* of the tree is then defined as

$$\operatorname{depth}(T) := \max_{t \in \mathcal{V}} \operatorname{level}(t) + 1\,.$$

(ii) A tree $T_I = (\mathcal{V}, \mathcal{E})$ is called *cluster tree* of a finite index set I if the following conditions hold:

(a) I is the root of T_I,

(b) for each vertex $t \in \mathcal{V}$ we have that t is a non-empty subset of I, its sons are pairwise disjoint, and $t = \bigcup_{t' \in \mathcal{S}(t)} t'$,

(c) all vertices which are not leaves have at least two sons.

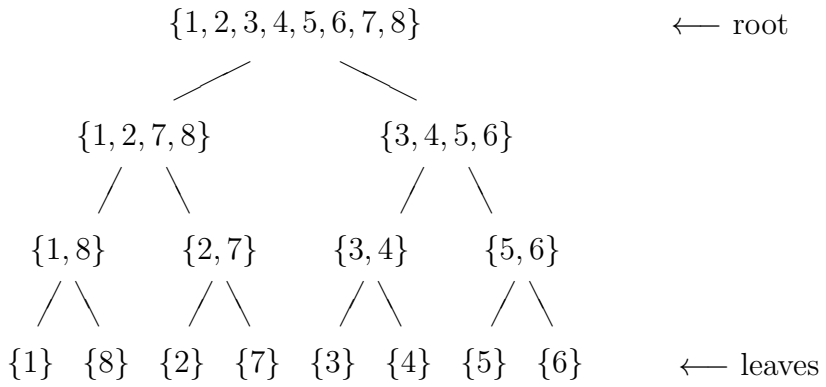


Figure 5.1: Example of a (binary) cluster tree.

Notation: For cluster trees we identify the tree with its vertices and write $t \in T_I$ instead of $t \in \mathcal{V}$.

(iii) We call a cluster tree T_I *balanced* if

$$R := \min_{t \in T_I \setminus \mathcal{L}(T_I)} \min \left\{ \frac{|t_1|}{|t_2|} : t_1,\, t_2 \in \mathcal{S}(t) \right\}$$

is bounded from below by a positive constant, independently of $|I|$.

Figure 5.1 shows an example of a cluster tree for the set $I = \{1, \ldots, 8\}$.

Lemma 5.15. *Let T_I be a balanced cluster tree. Then depth$(T_I) = \mathcal{O}(\log |I|)$. The storage complexity of T_I is then $\mathcal{O}(|I| \log |I|)$.*

For an index set I which represents a collection of elements $\{\tau_i\}_{i \in I}$, we want to generate a cluster tree T_I which will eventually lead to an admissible partition of $I \times I$ (or $I \times J$ for another cluster tree T_J). Here, we use the *principal component analysis* (PCA). To this end we associate to each element τ_i a point y_i (e. g., its center of gravity).

Definition 5.16. Let $t \subset I$ be a block.

- We define the *centroid* $m_t := \sum_{i \in t} \frac{|\tau_i|}{\sum_{j \in t} |\tau_j|}\, y_i$
- A vector $w_t \in \mathbb{R}^d$, $|w_t| = 1$ where the maximum

$$\max_{v \in \mathbb{R}^d,\, |v| = 1} \sum_{i \in t} |v \cdot (y_i - m_t)|^2$$

is attained is called *main direction* of t.

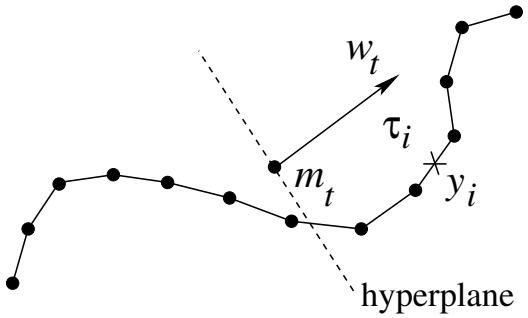


Figure 5.2: Example of block subdivision by the principal component analysis ($d = 2$).

- We define the *covariance matrix* $C_t := \sum_{i \in t} (y_i - m_t)\,(y_i - m_t)^\top \in \mathbb{R}^{d \times d}$.

Obviously, the computation of m_t can be done in linear time. But how to get the main direction? It is rather easy to see that w_t is a main direction of t if and only if $|w_t| = 1$ and w_t is an eigenvector to the maximal eigenvalue of C_t. Thus, in order to compute w_t we can form the covariance matrix C_t (in linear time) and compute its eigensystem.

Having m_t and w_t at our disposal we now subdivide the block t (which represents a collection of elements $\{\tau_i\}_{i \in t}$) using the hyperplane through m_t with normal w_t. We define the sons of t in the cluster tree by

$$\begin{aligned} \mathcal{S}(t) &:= \{t_1, t_2\} \\ t_1 &:= \{i \in t : w_t \cdot (y_1 - m_t) > 0\} \\ t_2 &:= t \setminus t_1\,, \end{aligned}$$

see also Figure 5.2. We apply this recursively to the set I and stop if a block contains less than $n_{\min}$ elements, where $n_{\min}$ is a fixed parameter. This way we get a cluster tree which we call *geometrically balanced*. Recall that we assume a shape-regular and quasi-uniform mesh. Then it can be shown that the cluster tree is also balanced in the sense of Definition 5.14(iii).

Lemma 5.17. *The construction of a cluster tree T_I for a collection of elements $\{\tau_i\}_{i \in I}$ using the principal component analysis as described above requires only $\mathcal{O}(|I| \log |I|)$ operations.*

5.7 Block Cluster Trees

We now use the cluster trees from the last section to construct an admissible partition for $I \times J$. Let T_I and T_J cluster trees for I and J, respectively, as

constructed above. We define the *block cluster tree* $T_{I\times J}$ by specifying the root to be $I \times J$ and defining for each block $t \times s$ the sons,

$$\mathcal{S}_{I\times J}(t\times s) := \begin{cases} \emptyset & \text{if } t\times s \text{ is admissible} \\ & \text{or } \mathcal{S}_I(t) = \emptyset \text{ or } \mathcal{S}_J(s) = \emptyset\,, \\ \mathcal{S}_I(t)\times\mathcal{S}_J(s) & \text{else.} \end{cases}$$

Above, $\mathcal{S}_{I\times J}$ indicates that the sons are to be understood with respect to the tree $T_{I\times J}$, and $\mathcal{S}_I$ with respect to the tree T_I etc. Obviously, by this construction,

$$\operatorname{depth}(T_{I\times J}) \;\le\; \min\{\operatorname{depth}(T_I), \operatorname{depth}(T_J)\}\,.$$

If T_I and T_J are binary trees (each vertex has two sons or is a leaf), $T_{I\times J}$ is a quadtree (each vertex has four sons or is a leaf).

Most importantly, by this construction the partition generated by the leaves of the block cluster tree $T_{I\times J}$ is admissible. This is because a leaf $t \times s$ is either admissible, or we have that t or s is itself a leaf of T_I or T_J, respectively, which implies that $\min(|t|, |s|) \le n_{\min}$, and so the block is small if we choose $n_{\min}$ accordingly.

A measure for the complexity of a block cluster tree is the so-called *sparsity constant*. Here, we introduce this concept only briefly, as we will just use it once on page 96. For a block $t \in T_I$ we define

$$c_{\text{sp}}^{\text{row}}(T_{I\times J},\, t) := \big|\{s \subset J : t\times s \in T_{I\times J}\}\big|\,,$$

i. e. the number of blocks $t \times s$ in the block cluster tree $T_{I\times J}$ with t being fixed. Similarly, for $s \in T_J$ we define

$$c_{\text{sp}}^{\text{col}}(T_{I\times J},\, s) := \big|\{t \subset I : t\times s \in T_{I\times J}\}\big|\,.$$

Finally, we define the sparsity constant of $T_{I\times J}$ by

$$c_{\text{sp}}(T_{I\times J}) := \max\Big\{ \max_{t\in T_I} c_{\text{sp}}^{\text{row}}(T_{I\times J},\, t)\,,\; \max_{s\in T_J} c_{\text{sp}}^{\text{col}}(T_{I\times J},\, s)\Big\}\,.$$

One can show that if T_I and T_J are geometrically balanced (e. g. constructed by the PCA) and the original mesh is shape-regular and quasi-uniform, the sparsity constant is bounded.

Lemma 5.18. *Let T_I and T_J be balanced cluster trees. Then the construction of the block cluster tree $T_{I\times J}$ as described above requires only $\mathcal{O}(|I|\log|I| + |J|\log|J|)$ operations.*

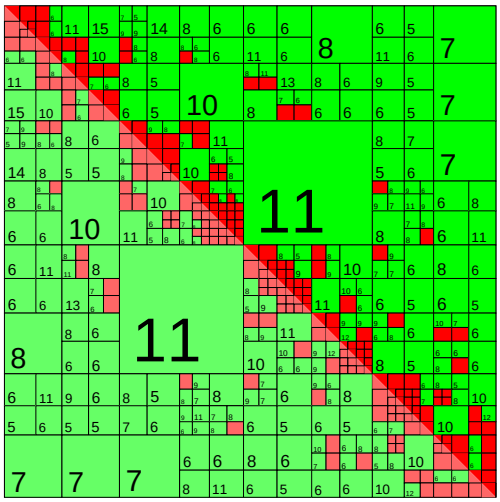


Figure 5.3: Example of a typical $\mathcal{H}$-matrix. Approximation of the single layer potential on a half-sphere discretized by 932 triangles. Green blocks: low-rank blocks (black numbers: local rank used by ACA). Red blocks: dense blocks.

5.8 The Set of Hierarchical Matrices

We will now work with the partition generated by our block cluster tree and define a special kind of matrix related to it.

Definition 5.19. The set of *hierarchichal matrices* on the block cluster tree $T_{I\times J}$ with an admissible partition $P = \mathcal{L}(T_{I\times J})$ and block-wise rank k is defined as

$$\mathcal{H}(T_{I\times J},\, k) := \left\{A \in \mathbb{R}^{I\times J} : \operatorname{rank}(A_{t\times s}) \leq k \quad \forall \text{ admissible blocks } t\times s \in P\right\}.$$

In short we call this set the set of $\mathcal{H}$-matrices.

An $\mathcal{H}$-matrix is stored as follows:

- for an admissible block $t \times s$ we use the outer product representation; the storage amount is then $k(|t| + |s|)$,
- for all other blocks we use the conventional entry-wise storate; the storage amount is then bounded by $n_{\min}(|t| + |s|)$.

A typical $\mathcal{H}$-matrix is shown in Figure 5.3 (for the ACA see the next section).

We now sketch how to prove that the total storage amount $N_{\mathrm{st}}(A)$ of an $\mathcal{H}$-matrix $A \in \mathcal{H}(T_{I\times J},\, k)$ fulfills

$$N_{\mathrm{st}} \;=\; \mathcal{O}\big(\max(k,\, n_{\min})\,(|I|\,\log|I| + |J|\,\log|J|)\big)\,.$$

In order to get the storage amount we have to sum up the individual storage amounts of each leave. In fact we will bound this by a sum over *all blocks* in the tree and use the sparsity constant (see page 94):

$$\begin{aligned}
N_{\mathrm{st}} \;&\le \sum_{t\times s\in\mathcal{L}(T_{I\times J})} \max(k,\, n_{\min})\,(|t| + |s|) \;\le \sum_{t\times s\in T_{I\times J}} \max(k,\, n_{\min})\,(|t| + |s|) \\
&\le\; \max(k,\, n_{\min}) \Big\{ \sum_{t\in T_I} \sum_{s\in T_J : t\times s\in T_{I\times J}} |t| + \sum_{s\in T_J} \sum_{t\in T_I : t\times s\in T_{I\times J}} |s| \Big\} \\
&\le\; c_{\mathrm{sp}}(T_{I\times J})\, \max(k,\, n_{\min}) \Big\{ \sum_{t\in T_I} |t| + \sum_{s\in T_J} |s| \Big\} \\
&\le\; \widetilde{C}\, \max(k,\, n_{\min})\,(|I|\,\log|I| + |J|\,\log|J|)\big)\,.
\end{aligned}$$

Many algorithms in the context of $\mathcal{H}$-matrices perform the same kind of operations blockwise. The above technique can be used to estimate the total complexity of these algorithms.

Remark 5.20. $\mathcal{H}$-matrices generalize sparse matrices. One can e. g. show that a FEM stiffness matrix can be stored as an $\mathcal{H}$-matrix even with $\mathcal{O}(n)$ storage requirements. In constrast to sparse matrices, we call $\mathcal{H}$-matrices *data-sparse*.

Lemma 5.21. *The computational complexity of the $\mathcal{H}$-matrix by vector multiplication is $\mathcal{O}\big(\max\{k,\, n_{min}\}\big(|I|\,\log|I| + |J|\,\log|J|\big)\big)$.*

This means that we can not only use $\mathcal{H}$-matrices to represent approximations of BEM matrices, but also for iterative solver, as for instance CG, where only the application of the matrix to a vector is required. To get quasi-optimal solvers one needs of course preconditioners, and one can in fact generalize the known multi-level preconditioners for FEM (such as BPX type preconditioners) also for this purpose.

However, we will not discuss that issue, but instead point out that a whole $\mathcal{H}$-arithmetic is available, similar to the floating point arithmetic in processors. E. g. we can add two $\mathcal{H}$-matrices which are defined on the same block cluster tree by using conventional addition in the dense blocks and rounded addition in the low-rank blocks. This way, we obtain again an $\mathcal{H}$-matrix which is an approximation of the exact sum. Multiplication is rather straigtforward and can be based on the multiplication and rounded addition

of low-rank matrices. With the following idea one can even approximate the *inverse* of an $\mathcal{H}$-matrix by an $\mathcal{H}$-matrix. Consider a block t split into two sub-blocks t_1 and t_2 and suppose that

$$A_{tt} = \begin{bmatrix} A_{t_1t_1} & A_{t_1t_2} \\ A_{t_2t_1} & A_{t_2t_2} \end{bmatrix}.$$

Then by block-elimination, we have

$$A_{tt}^{-1} = \begin{bmatrix} A_{t_1t_1}^{-1} + A_{t_1t_1}^{-1} A_{t_1t_2} S^{-1} A_{t_2t_1} A_{t_1t_1}^{-1} & -A_{t_1t_1}^{-1} A_{t_1t_2} S^{-1} \\ -S^{-1} A_{t_2t_1} A_{t_1t_1}^{-1} & S^{-1} \end{bmatrix},$$

with the Schur complement $S := A_{t_2t_2} - A_{t_2t_1} A_{t_1t_1}^{-1} A_{t_1t_2}$. Using rounded addition, multiplication and recursive $\mathcal{H}$-inverse approximations for all the involved operations, one reaches some point where blocks are small and where one can use and form the conventional inverses exactly. With the same idea one can derive an $\mathcal{H}$-*LU* decomposition.

The following lemma summarizes the computational complexities.

Lemma 5.22. *1. Rounded addition of two $\mathcal{H}$-matrices requires*

$$\mathcal{O}(|I| \log |I| + |J| \log |J|)$$

operations.

2. *For $I = J$, rounded multiplication of two $\mathcal{H}$-matrices requires*

$$\mathcal{O}(k^2 |I| \log |I| + k^3 |I|)$$

operations.

3. *For $I = J$, the $\mathcal{H}$-inverse of an $\mathcal{H}$-matrix can be formed in*

$$\mathcal{O}(k^2 |I| \log |I| + k^3 |I|)$$

operations.

4. *For $I = J$, the $\mathcal{H}$-LU decomposition of an $\mathcal{H}$-matrix can be computed in*

$$\mathcal{O}(k^2 |I| \log^2 |I|)$$

operations. Once the decomposition is computed, solving can be done by forward-backward substitution.

5.9 The Adaptive Cross Approximation (ACA)

Prescribing a maximal rank and using the Taylor expansion, on could approximate our BEM matrices by suitable $\mathcal{H}$-matrices. However, (i) an error control is hard to be achieved, and (ii) we have to write new code for the integrals due to the Taylor expansion. In this final section we describe an algorithm where one can reuse existing code and where one has error control.

In order to introduce this technique we formally run the following algorithm for a matrix $A \in \mathbb{R}^{m\times n}$.

$R_0 := A$

For $\ell = 0, 1, \ldots$

Find a non-zero pivot-element (i_ℓ, j_ℓ)

$$R_{\ell+1} := R_\ell - \frac{1}{[R_\ell]_{i_\ell j_\ell}} (R_\ell)_{1:m,j_\ell} (R_\ell)_{i_\ell,1:n}$$

Until ?

Here, $(R_\ell)_{1:m,j_\ell}$ denotes the j_ℓ-th column and $(R_\ell)_{i_\ell,1:n}$ the i_ℓ-th row of R_ℓ.

Example 5.23.

$$R_0 = \begin{bmatrix} 0.431 & 0.345 & \mathbf{0.582} & 0.417 & 0.455 \\ 0.491 & 0.396 & 0.674 & 0.449 & 0.427 \\ 0.446 & 0.358 & 0.583 & 0.413 & 0.441 \\ 0.380 & 0.328 & 0.557 & 0.372 & 0.349 \\ 0.412 & 0.340 & 0.516 & 0.375 & 0.370 \end{bmatrix} \quad \begin{matrix} i_1=1 \\ j_1=3 \\ \to \end{matrix} \quad \frac{1}{0.582} \begin{bmatrix} 0.582 \\ 0.674 \\ 0.583 \\ 0.557 \\ 0.516 \end{bmatrix} \begin{bmatrix} 0.431 \\ 0.354 \\ 0.582 \\ 0.417 \\ 0.455 \end{bmatrix}^\top$$

$$R_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.008 & -0.014 & 0 & -0.033 & \mathbf{-0.100} \\ 0.014 & 0.003 & 0 & -0.004 & -0.014 \\ -0.032 & -0.011 & 0 & -0.026 & -0.087 \\ 0.029 & 0.025 & 0 & 0.005 & -0.034 \end{bmatrix} \quad \begin{matrix} i_2=2 \\ j_2=5 \\ \to \end{matrix} \quad \frac{1}{-0.1} \begin{bmatrix} 0 \\ -0.100 \\ -0.014 \\ -0.087 \\ -0.034 \end{bmatrix} \begin{bmatrix} -0.008 \\ -0.014 \\ 0 \\ -0.033 \\ -0.100 \end{bmatrix}^\top$$

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbf{0.016} & 0.005 & 0 & 0.000 & 0 \\ -0.020 & 0.001 & 0 & 0.002 & 0 \\ 0.032 & 0.030 & 0 & 0.017 & 0 \end{bmatrix} \quad \begin{matrix} i_3=3 \\ j_3=1 \\ \to \end{matrix} \quad \frac{1}{0.016} \begin{bmatrix} 0 \\ 0 \\ 0.016 \\ -0.020 \\ 0.032 \end{bmatrix} \begin{bmatrix} 0.016 \\ 0.005 \\ 0 \\ 0.000 \\ 0 \end{bmatrix}^\top$$

In the (only illustrative) example we see that we formally subtract a "cross" build by the row and column we have chosen. Also, the absolute values of the remaining matrix entries get smaller and smaller.

Suppose that we stop the iteration after k steps, we have

$$A \;=\; S_k + R_k\,, \qquad \text{with} \qquad S_k \;=\; \sum_{\ell=1}^{k} u_\ell\, v_\ell^\top\,.$$

The matrix S_k (which is naturally provided in outer-product representation) will be used as an approximation. Obviously, for the computation of S_k we only need to know *some* matrix entries *on demand*, i. e., we can reuse existing code.

The remaining two questions are how to chose the pivot elements and when to stop. We will only answer partially them. There exist algorithms to choose the right pivot elements such that when we use the stopping criterion

$$|u_\ell| \, |v_\ell| \;\leq\; \frac{\varepsilon \, (1-\eta)}{1+\varepsilon} \, \|S_k\|_F$$

for a prescribed tolerance ε and the parameter η from the admissibility condition, we get the guaranteed error bound

$$\|A - S_k\|_F \;\leq\; \varepsilon \, \|A\|_F \, .$$

The total computational complexity of the approximation is

$$\mathcal{O}(n \, \log n \, |\log \varepsilon|^{2d})$$

with $n = |I| \sim |J|$ for building the approximation. The storage amount and the complexity of the matrix-vector multiplication is

$$\mathcal{O}(n \, \log n \, |\log \varepsilon|^{d}) \, .$$

The ACA can also used for preconditioning (by choosing ε large, e. g., 0.1).

Software packages:

- `Hlib`, see `www.hlib.org`
- `AHMED` (by Mario Bebendorf)

Further reading: [Bebendorf].

Chapter 6

FEM-BEM Coupling*

For simplicity, we consider the coupling of a three-dimensional interior domain Ω^{int} and the corresponding exterior domain $\Omega^{\text{ext}} = \mathbb{R}^3 \setminus \overline{\Omega}^{\text{int}}$ with interface Γ, see Figure 6.1, left.

The global equation in distributional form is

$$\text{find } u \in H^1_E(\mathbb{R}^3): \quad -\text{div}(\alpha \nabla u) + c\, u \;=\; f \qquad \text{in } \mathcal{D}^*(\mathbb{R}^3), \tag{6.1}$$

where

$$f \in L^2(\mathbb{R}^3), \qquad f_{|\Omega^{\text{ext}}} = 0,$$

and α, $c \in L^\infty(\mathbb{R}^3)$ fulfilling

$$\begin{aligned} &\alpha(x) \geq \alpha_0 > 0, && c(x) \geq c_0 > 0 \qquad \forall x \in \Omega^{\text{int}} \text{ a.e.} \\ &\alpha_{|\Omega^{\text{ext}}} = \alpha^{\text{ext}} = \text{const} > 0, && c_{|\Omega^{\text{ext}}} = 0. \end{aligned}$$

In the above situation, neither the FEM alone is suitable (because $\mathbb{R}^3$ is unbounded), nor the BEM (because α and/or c is non-constant).

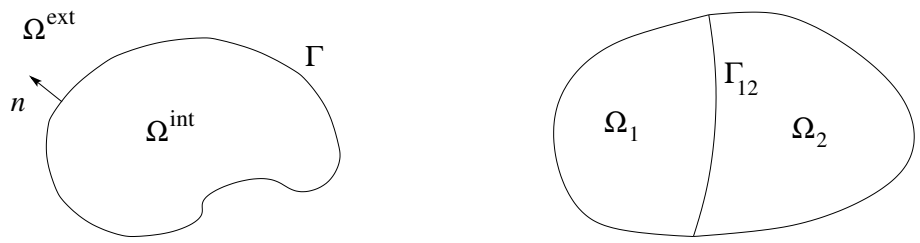


Figure 6.1: *Left:* Ω^{int}, Ω^{ext}, interface Γ. *Right:* Bounded domain Ω splits into two subdomains Ω_1, Ω_2.

Problem (6.1) is equivalent to

$$\text{find } u \in H^1_E(\mathbb{R}^3): \quad \int_{\mathbb{R}^3} \alpha\nabla u\cdot\nabla v + c\,u\,v\,dx \;=\; \int_{\Omega^{\text{int}}} f\,v\,dx \qquad \forall v \in H^1_E(\mathbb{R}^3).$$

Thanks to the definition of the exterior normal derivative γ_1^{ext}, we have that

$$\int_{\Omega^{\text{ext}}} \alpha^{\text{ext}}\nabla u\cdot\nabla v\,dx \;=\; -\langle\gamma_1^{\text{ext}}u,\,\gamma_0 v\rangle_\Gamma \qquad \forall v \in H^1_E(\Omega^{\text{ext}}).$$

Therefore, we can *rewrite* (6.1) as

$$\begin{aligned} &\text{find } u \in H^1(\Omega^{\text{int}}): \qquad (6.2)\\ &\underbrace{\int_{\Omega^{\text{int}}} \alpha\nabla u\cdot\nabla v + c\,u\,v\,dx}_{=:\,a^{\text{int}}(u,\,v)} - \langle t,\,\gamma_0^{\text{int}}v\rangle_\Gamma \;=\; \int_{\Omega^{\text{int}}} f\,v\,dx \qquad \forall v \in H^1(\Omega^{\text{int}}), \end{aligned}$$

where $t = \gamma_1^{\text{ext}}u^{\text{ext}}$ is the exterior conormal derivative of $u^{\text{ext}} \in H^1_E(\Omega^{\text{ext}})$, fulfilling

$$\begin{aligned} -\alpha^{\text{ext}}\Delta u^{\text{ext}} &= 0 && \text{in } \mathcal{D}^*(\Omega^{\text{ext}}),\\ \gamma_0^{\text{ext}}u^{\text{ext}} &= \gamma_0^{\text{int}}u && \text{in } H^{1/2}(\Gamma). \end{aligned} \qquad (6.3)$$

Remark 6.1. 1. The restrictions $d = 2$ and $c_0 \geq 0$ were chosen to avoid certain technicalities, the coupling formulations below, however, can essentially be used also for the more general differential operator $Lu = -\text{div}(\alpha\nabla u) + c\,u$ and also for $d = 2$.

2. A similar coupling problem is the following. Let Ω be a bounded (weakly) Lipschitz domain that splits into two disjoint parts Ω_1, Ω_2 (again Lipschitz) such that

$$\overline{\Omega} \;=\; \overline{\Omega}_1 \cup \overline{\Omega}_2$$

with the interface $\Gamma_{12} := \partial\Omega_1 \cap \partial\Omega_2$, see Fig. 6.1, and with the usual PDE posed in Ω with suitable boundary conditions. In that case, a similar formulation to (6.2)–(6.3) can be derived.

6.1 Non-symmetric Coupling

Since (6.3) is an exterior Dirichlet problem, we obtain from Sect. 3.6.4 that the conormal derivative $t \in H^{-1/2}(\Gamma)$ fulfills

$$Vt \;=\; (-\tfrac{1}{2}I + K)\gamma_0^{\text{int}}u.$$

Rewriting this boundary integral equation in variational form, we obtain together with (6.2) the mixed variational problem

$$\begin{aligned} &\text{find } (u,\, t) \in H^1(\Omega^{\text{int}}) \times H^{-1/2}(\Gamma) : \qquad (6.4)\\ &\qquad a^{\text{int}}(u,\, v) - \langle t,\, \gamma_0^{\text{int}} v\rangle_\Gamma \;=\; \int_{\Omega^{\text{int}}} f\, v\, dx \qquad \forall v \in H^1(\Omega^{\text{int}}),\\ &\langle \tau,\, (\tfrac{1}{2} I - K)\gamma_0^{\text{int}} u\rangle_\Gamma + \langle \tau,\, Vt\rangle_\Gamma \;=\; 0 \qquad \forall \tau \in H^{-1/2}(\Gamma). \end{aligned}$$

Adding up the two equations leads to a non-symmetric bilinear form $\mathcal{B} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ with $\mathcal{V} := H^1(\Omega^{\text{int}}) \times H^{-1/2}(\Gamma)$.

Remark 6.2. The following result can be shown (cf. [Steinbach2011]). If $\alpha(x) \geq \alpha_0 > \frac{1}{4}\,\alpha^{\text{ext}}$ for almost every $x \in \Omega^{\text{int}}$, then $\mathcal{B}$ is elliptic on $\mathcal{V}$ with respect to the norm $(\|u\|^2_{H^1(\Omega^{\text{int}})} + \|t\|^2_{H^{-1/2}(\Gamma)})^{1/2}$.

Assume that Ω is polyhedral and consider a triangulation $\mathcal{T}_h(\Omega^{\text{int}})$. We discretize the variable u with FEM using the space

$$S_h^1(\Omega^{\text{int}}) := \{v \in C(\Omega^{\text{int}}) : v_{|\tau} \in \mathcal{P}^1 \quad \forall \tau \in \mathcal{T}_h(\Omega^{\text{int}})\}$$

of continuous and piecewise linear functions. The volume triangulation $\mathcal{T}_h(\Omega^{\text{int}})$ naturally induces a surface triangulation $\mathcal{T}_h(\Gamma)$ (by restriction). Using the space

$$S_h^0(\Gamma) = \{w \in L^2(\Gamma) : w_{|\tau} \in \mathcal{P}^0 \quad \forall \tau \in \mathcal{T}_h(\Gamma)\},$$

we can discretize t.

The Galerkin discretization of (6.4) using the two spaces above reads

$$\begin{aligned} &\text{find } (u_h,\, t_h) \in S_h^1(\Omega^{\text{int}}) \times S_h^0(\Gamma) : \qquad (6.5)\\ &\qquad a^{\text{int}}(u_h,\, v_h) - \int_\Gamma t_h\, v_h\, ds \;=\; \int_{\Omega^{\text{int}}} f\, v_h\, dx \quad \forall v_h \in S_h^1(\Omega^{\text{int}}),\\ &\langle \tau_h,\, (\tfrac{1}{2} I - K)\gamma_0^{\text{int}} u_h\rangle_\Gamma + \langle \tau_h,\, Vt_h\rangle \;=\; 0 \qquad \forall \tau_h \in S_h^0(\Gamma). \end{aligned}$$

Let A_h be the FE stiffness matrix corresponding to $a^{\text{int}}(\cdot,\cdot)$ with respect to a basis of $S_h^1(\Omega^{\text{int}}) \subset H^1(\Omega^{\text{int}})$ (e.g., the common hat functions). Then (6.5) takes the form

$$\begin{bmatrix} A_h & M_h \\ \frac{1}{2}M_h - K_h & V_h \end{bmatrix} \begin{bmatrix} \underline{u}_h \\ \underline{t}_h \end{bmatrix} \;=\; \begin{bmatrix} \underline{f}_h \\ 0 \end{bmatrix}. \qquad (6.6)$$

This formulation is often used by engineers.

Remark 6.3. If $\mathcal{B}$ above is elliptic (see Remark 6.2), its Galerkin discretization (6.8) is uniquely solvable. Stability and convergence follow from Céa's lemma and from the error estimates for $S_h^1(\Omega^{\text{int}})$ and $S_h^0(\Gamma)$.

6.2 Symmetric Coupling

In contrast to the previous section, we use the *complete* Calderón equations in Ω^{ext}: the Cauchy data $(\gamma_0^{\text{int}} u,\, t)$ fulfill

$$\begin{aligned} Vt + (\tfrac{1}{2}I - K)\gamma_0^{\text{int}} u &= 0, \\ t &= -D\gamma_0^{\text{int}} u + (\tfrac{1}{2}I - K')t. \end{aligned}$$

Substituting the second identity into the term $-\langle t,\, \gamma_0^{\text{int}} v\rangle_\Gamma$, we obtain the following problem.

$$\begin{aligned} &\text{Find } (u,\, t) \in H^1(\Omega^{\text{int}}) \times H^{-1/2}(\Gamma) : \qquad (6.7) \\ &\quad a^{\text{int}}(u,\, v) + \langle D\gamma_0^{\text{int}} u,\, \gamma_0^{\text{int}} v\rangle_\Gamma - \langle (\tfrac{1}{2}I - K')t,\, \gamma_0^{\text{int}} v\rangle_\Gamma = \int_{\Omega^{\text{int}}} f\, v\, dx \\ &\quad \langle \tau,\, (\tfrac{1}{2}I - K)\gamma_0^{\text{int}} u\rangle_\Gamma + \langle \tau,\, Vt\rangle_\Gamma = 0 \\ &\qquad\qquad \forall v \in H^1(\Omega^{\text{int}}) \quad \forall \tau \in H^{-1/2}(\Gamma). \end{aligned}$$

When multiplying the second equation by -1, the corresponding bilinear form becomes symmetric.

Exercise 28. Show that $\mathcal{S}((u,\, t),\, (v,\, \tau)) = \int_{\Omega^{\text{int}}} \alpha \nabla u \cdot \nabla v + c\, u\, v\, dx + \langle D\gamma_0^{\text{int}} u,\, \gamma_0^{\text{int}} v\rangle_\Gamma - \langle (\tfrac{1}{2}I - K')t,\, \gamma_0^{\text{int}} v\rangle_\Gamma + \langle \tau,\, (\tfrac{1}{2}I - K)\gamma_0^{\text{int}} u\rangle_\Gamma + \langle \tau,\, Vt\rangle_\Gamma$ is a bounded and elliptic bilinear form on $\mathcal{V}$ with respect to the norm $(\|u\|^2_{H^1(\Omega)} + \|t\|^2_{H^{-1/2}(\Gamma)})^{1/2}$.

Using the analogous discretization as in Sect. 6.1, we obtain a linear system of the form

$$\begin{bmatrix} A_h + D_h & -\frac{1}{2}M_h^\top + K_h^\top \\ \frac{1}{2}M_h - K_h & V_h \end{bmatrix} \begin{bmatrix} \underline{u}_h \\ \underline{t}_h \end{bmatrix} = \begin{bmatrix} \underline{f}_h \\ 0 \end{bmatrix}. \qquad (6.8)$$

Multiplying the last line by -1 yields a *symmetric* (indefinite) system matrix. As in the previous section, convergence follows from Céa's lemma and the approximation estimates for $S_h^1(\Omega^{\text{int}})$ and $S_h^0(\Gamma)$.

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