# Super Question for the Oral Examination

Piezoelectrical materials show electrical effects under mechanical loading and, vice versa, mechanical deformations by applying some voltage. The direct and the inverse piezoelectrical effects are nowadays used in many applications. The magnetic valves in our cars were replaced by the more efficient piezoelectrical valves some years ago. The SAWs (Surface Acoustic Wavers) are the most important part in our mobiles. So, the simulation and, of course, the optimization of the piezoelectrical behavior is very important in the electronic industry. The stationary piezoelectrical behavior can be described by the following variational problem: Find $(u, \varphi) \in V_g = V_0 = D \times P$ such that

$$a((u,\varphi),(v,\psi)) = \langle (F,G),(v,\psi)\rangle \quad \forall (v,\psi) \in V_0, \tag{1}$$

where

- $u = (u_1, u_2, u_3)$ - displacement field,
- $\varphi$ - electrical potential ($E = -\nabla\varphi$ = electrical field),
- $D = \{v \in (H^1(\Omega))^3 : v = 0 \text{ on } \Gamma_{1,u}\}$, $\text{meas}\Gamma_{1,u} > 0$,
- $P = \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_{1,\varphi}\}$, $\text{meas}\Gamma_{1,\varphi} > 0$,
- $a((u,\varphi),(v,\psi)) = c(u,v) + b(\varphi,v) - b(\psi,u) + k(\psi,\varphi)$,
  where (Einstein's summation convention)
  $c(u,v) = \int_\Omega \varepsilon(u) : C : \varepsilon(v) dx = \int_\Omega \varepsilon_{ij}(u) C_{ijkl} \varepsilon_{kl}(v)\, dx$,
  $b(\psi,v) = \int_\Omega \varepsilon(v) : B \cdot \nabla\psi\, dx = \int_\Omega \varepsilon_{ij}(v) B_{ijk} \frac{\partial\psi}{\partial x_k}\, dx$,
  $k(\varphi,\psi) = \int_\Omega \nabla\varphi \cdot K \cdot \nabla\psi\, dx = \int_\Omega \frac{\partial\varphi}{\partial x_i} K_{ij} \frac{\partial\psi}{\partial x_k}\, dx$,
  with the strains $\varepsilon_{ij}(u) = 0.5(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, the material tensors $C$ (symmetric tensor of the elastic coefficients), $B$ (piezoelectrical coupling tensor: $B_{ijk} = B_{jik}\, \forall i,j,k = 1,2,3$), $K$ (symmetric dielectrical material tensor) possess the following properties: $C_{ijkl}, B_{ijk}, K_{ij} \in L_\infty(\Omega)$ and there exist positive constants $\underline{c}, \bar{c}, \underline{k}$ and $\bar{k}$ such that
  1. $\underline{c}\, \varepsilon_{ij}\varepsilon_{ij} \le \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \le \bar{c}\, \varepsilon_{ij}\varepsilon_{ij} \quad \forall \varepsilon = (\varepsilon_{ij}) \in R^9_{sym}$,
  2. $\underline{k}\, \xi_i\xi_i \le \xi_i K_{ij} \xi_j \le \bar{k}\, \xi_i\xi_i \quad \forall \xi = (\xi_i) \in R^3$,
- $\langle (F,G),(v,\psi)\rangle = \langle F, v\rangle + \langle G, \psi\rangle$,
  where (Einstein's summation convention)
  $\langle F, v\rangle = \int_\Omega f \cdot v\, dx + \int_{\Gamma_{2,u}} t \cdot v\, ds = \int_\Omega f_i v_i\, dx + \int_{\Gamma_{2,u}} t_i v_i\, ds$,
  $\langle G, \psi\rangle = \int_\Omega \varrho\psi\, dx + \int_{\Gamma_{2,\varphi}} \omega\psi\, ds$,
  with given functions $f_i, \varrho \in L_2(\Omega)$, $t_i \in L_2(\Gamma_{2,u})$ and $\omega \in L_2(\Gamma_{2,\varphi})$.
- $\Omega \subset R^3$ is a bounded Lipschitz domain.

Solve the following tasks:

1. Derive the classical formulation of the piezoelectrical boundary value problem from its variational formulation (1) !

2. Show existence and uniqueness of the solution $(u, \varphi)$ of piezoelectrical variational problems (1) !

3. Derive the residual a posteriori error estimator for the discretization error

$$\|(u, \varphi) - (u_h, \varphi_h)\|_{D\times P} = \sqrt{\|u - u_h\|^2_{H^1(\Omega)} + \|\varphi - \varphi_h\|^2_{H^1(\Omega)}} \leq ? \tag{2}$$

   in the same way as we did it in our lecture for the homogeneous Dirichlet problem for the Poisson equation, where $(u_h, \varphi_h) \in V_{gh} = V_{0h} = D_h \times P_h$ is the finite element solution to (1), i.e.

$$a((u_h, \varphi_h), (v_h, \psi_h)) = \langle (F, G), (v_h, \psi_h) \rangle \quad \forall\ (v_h, \psi_h) \in V_{0h} = D_h \times P_h, \tag{3}$$

   where the finite element discretization is obtained by continuous, piecewise linear tetrahedral elements on regular tetrahedral triangulation $\tau_h$ of the computational domain $\Omega$ !

**Hint:**
Beside triangle, Cauchy's and Friedrichs' inequalities you need also the following version of Korn's second inequality: There exists a positive constant $c_K$ such that the inequality (Einstein's notation)

$$\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\, dx \geq c_K \|v\|^2_{H^1(\Omega)}$$

holds for all $v \in D = \{v \in (H^1(\Omega))^3 : v = 0 \text{ on } \Gamma_{1,u}\}$.