1 Interpolation in $\mathbb{R}^n$

Matrix functions

Needed here only for diagonalizable matrices: Let M be of the form

$$M = XDX^{-1} \quad \text{with} \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and a non-singular matrix X. Observe that λ_i are the eigenvalues of M and the columns of X are eigenvectors of M. Let f be a scalar function well-defined on the spectrum of M. Then we set

$$f(M) = X^{-1}f(D)X \quad \text{with} \quad f(D) = \begin{bmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{bmatrix}.$$

$f(M)$ is well-defined.

In particular, we have: For each symmetric and positive definite matrix M:

$$\left(M^{1/2}\right)^2 = M.$$

Inner products and symmetric and positive definite matrices

Let $X = (\mathbb{R}^n, (\cdot,\cdot)_X)$ be a Hilbert space, where $(\cdot,\cdot)_X$ denotes an inner product in $\mathbb{R}^n$. Then

$$(x, y)_X = \sum_{i,j=1}^{n} (e_i, e_j)x_i y_j,$$

where $\{e_i \in \mathbb{R}^n \colon i = 1, \ldots, n\}$ denotes the natural basis in $\mathbb{R}^n$. The matrix M, given by

$$M = (M_{ij}) \quad \text{with} \quad M_{ij} = (e_i, e_j)_X,$$

is symmetric and positive definite and we have

$$(x, y)_X = y^T Mx = \langle Mx, y\rangle$$

with the Euclidean inner product

$$\langle x, y\rangle = y^T x.$$

On the other hand, let M be a symmetric and positive definite matrix. Then this matrix defines an inner product:

$$(x, y)_X = \langle Mx, y\rangle$$

leading to a Hilbert space $X = (\mathbb{R}^n, (\cdot,\cdot)_X)$.

The dual norm

Let $f \in \mathbb{R}^n$. For the dual norm

$$\|f\|_{X^*} = \sup_{x \in X} \frac{\langle f, x\rangle}{\|x\|_X}$$

we have

$$\begin{aligned}\|f\|_{X^*}^2 &= \sup_{x \in X} \frac{\langle f, x\rangle^2}{\langle Mx, x\rangle} = \sup_{x \in X} \frac{\langle f, M^{-1/2}x\rangle^2}{\langle x, x\rangle} = \sup_{x \in X} \frac{\langle M^{-1/2}f, x\rangle^2}{\langle x, x\rangle} \\ &= \langle M^{-1/2}f, M^{-1/2}f\rangle = \langle M^{-1}f, f\rangle\end{aligned}$$

leading to the dual Hilbert space $X^* = (\mathbb{R}^n, (\cdot,\cdot)_{X^*})$. So M^{-1} represents the dual norm.

Sum and intersection

Let M_0 and M_1 be two symmetric and positive definite matrices. These matrices define two inner products:

$$(x, y)_{X_0} = \langle M_0 x, y\rangle, \quad (x, y)_{X_1} = \langle M_1 x, y\rangle$$

leading to two Hilbert spaces $X_0 = (\mathbb{R}^n, (\cdot,\cdot)_{X_0})$ and $X_1 = (\mathbb{R}^n, (\cdot,\cdot)_{X_1})$.

The norms of $X_0 + X_1$ and $X_0 \cap X_1$ (sum and the intersection of the Hilbert spaces X_0, X_1) are given by

$$\|x\|_{X_0+X_1}^2 = \inf_{x=x_0+x_1} \left(\|x_0\|_{X_0}^2 + \|x_1\|_{X_1}^2\right), \quad \|x\|_{X_0\cap X_1}^2 = \|x\|_{X_0}^2 + \|x\|_{X_1}^2.$$

It is easy to see that these norms are the associated norms to the following inner products

$$(x, y)_{X_0+X_1} = \langle \left(M_0^{-1} + M_1^{-1}\right)^{-1} x, y\rangle, \quad (x, y)_{X\cap X_1} = \langle (M_0 + M_1)\, x, y\rangle. \tag{1}$$

So, we have introduced the Hilbert spaces $X_0 + X_1 = (\mathbb{R}^n, (\cdot,\cdot)_{X_0+X_1})$ and $X_0 \cap X_1 = (\mathbb{R}^n, (\cdot,\cdot)_{X_0\cap X_1})$.

From (1) it follows that

$$(x, y)_{(X_0+X_1)^*} = (x, y)_{X_0^* \cap X_1^*}$$

and

$$(x, y)_{(X_0\cap X_1)^*} = (x, y)_{X_0^* + X_1^*}.$$

Hence

$$(X_0 + X_1)^* = X_0^* \cap X_1^* \quad \text{and} \quad (X_0 \cap X_1)^* = X_0^* + X_1^*.$$

Observe that

$$\frac{1}{\sqrt{2}} \inf_{x=x_1+x_2} (\|x_0\|_{X_0} + \|x_1\|_{X_1}) \le \|x\|_{X_0+X_1} \le \inf_{x=x_1+x_2} (\|x_0\|_{X_0} + \|x_1\|_{X_1})$$

and

$$\max(\|x\|_{X_0}, \|x\|_{X_1}) \le \|x\|_{X_0\cap X_1} \le \sqrt{2}\max(\|x\|_{X_0}, \|x\|_{X_1})$$

Moreover,

$$\|x\|_{X_0+X_1} \le \min(\|x\|_{X_0}, \|x\|_{X_1}).$$

A generalized eigenvalue problem

Consider the generalized eigenvalue problem:

$$M_1 x = \lambda\, M_0 x,$$

or, equivalently;

$$M_0^{-1} M_1 x = \lambda\, x.$$

Observe that $M_0^{-1}M_1$ is self-adjoint and positive definite with respect to the inner product $(\cdot,\cdot)_{X_0}$. Therefore, there is a basis $\{e_i \colon i = 1, 2, \dots, n\}$ of eigenvectors with

$$(e_i, e_j)_{X_0} = \delta_{ij}.$$

Each vector $x \in \mathbb{R}^n$ can be written in the form

$$x = \sum_{i=1}^{n} \hat{x}_i\, e_i.$$

Then

$$\|x\|_{X_0}^2 = \sum_{i=1}^{n} \hat{x}_i^2 \quad \text{and} \quad \|x\|_{X_1}^2 = \sum_{i=1}^{n} \lambda_i\, \hat{x}_i^2.$$

Moreover,

$$\|x\|_{X_0+X_1}^2 = \sum_{i=1}^{n} (1+\lambda_i^{-1})^{-1}\, \hat{x}_i^2 = \sum_{i=1}^{n} \frac{\lambda_i}{1+\lambda_1}\, \hat{x}_i^2$$

and

$$\|x\|_{X_0\cap X_1}^2 = \sum_{i=1}^{n} (1+\lambda_i)\, \hat{x}_i^2.$$

A first part of the interpolation theorem

Theorem 1.1. *Let $T \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with*

$$\|Tx\|_{Y_0} \le c_0\, \|x\|_{X_0} \quad \textit{and} \quad \|Tx\|_{Y_1} \le c_1\, \|x\|_{X_1},$$

where the norms $\|\cdot\|_{X_i}$ and $\|\cdot\|_{Y_i}$ are the norms associated to the inner products :

$$(x, y)_{X_i} = \langle M_i x, y\rangle \quad \textit{and} \quad (x, y)_{Y_i} = \langle N_i x, y\rangle \quad \textit{for } i = 0, 1.$$

Then,

$$\|Tx\|_{Y_0+Y_1} \le \max(c_0, c_1)\, \|x\|_{X_0+X_1}$$

and

$$\|Tx\|_{Y_0\cap Y_1} \le \max(c_0, c_1)\, \|x\|_{X_0\cap X_1}.$$

Proof. We have

$$\begin{aligned}
\|Tx\|_{Y_0+Y_1}^2 &= \inf_{Tx=y_0+y_1} \left(\|y_0\|_{Y_0}^2 + \|y_1\|_{Y_1}^2\right) \\
&\le \inf_{x=x_0+x_1} \left(\|Tx_0\|_{Y_0}^2 + \|Tx_1\|_{Y_1}^2\right) \\
&\le \inf_{x=x_0+x_1} \left(c_0^2\,\|x_0\|_{X_0}^2 + c_1^2\,\|x_1\|_{X_1}^2\right) \\
&\le \max(c_0^2, c_1^2) \inf_{x=x_0+x_1} \left(\|x_0\|_{X_0}^2 + \|x_1\|_{X_1}^2\right) \\
&= \max(c_0^2, c_1^2)\,\|x\|_{X_0+X_1}^2.
\end{aligned}$$

The proof of the second estimate is straight forward. □

The norm $\|\cdot\|_\theta$

For each $\theta \in [0,1]$ we introduce a norm by

$$\|x\|_\theta^2 = \sum_{i=1}^n \lambda_i^\theta\,\hat{x}_i^2.$$

Observe that

$$\|x\|_0 = \|x\|_{X_0} \quad \text{and} \quad \|x\|_1 = \|x\|_{X_1}.$$

So, we have introduced the normed space $X_\theta = (\mathbb{R}^n, \|\cdot\|_\theta)$. We will also use that notation $\|x\|_{X_\theta}$ for $\|x\|_\theta$.

A first representation of the norm $\|\cdot\|_\theta$

We have

$$(M_0^{-1/2} M_1 M_0^{-1/2})(M_0^{1/2} e_i) = \lambda_i\,(M_0^{1/2} e_i).$$

Therefore,

$$(M_0^{-1/2} M_1 M_0^{-1/2})^\theta (M_0^{1/2} e_i) = \lambda_i^\theta\,(M_0^{1/2} e_i),$$

which implies

$$\left\langle (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2} e_i, M_0^{1/2} e_j \right\rangle = \lambda_i^\theta \left\langle M_0^{1/2} e_i, M_0^{1/2} e_j \right\rangle = \lambda_i^\theta\,\delta_{ij}.$$

By multiplying with $\hat{x}_i$ and $\hat{x}_j$ and summing over i and j we obtain

$$\sum_{i=1}^n \lambda_i^\theta\,\hat{x}_i^2 = \left\langle (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2} x, M_0^{1/2} x \right\rangle$$

This shows that the norm $\|\cdot\|_{X_\theta}$ is the associated norm to the inner product

$$(x,y)_{X_\theta} = \langle M_\theta x, y\rangle \quad \text{with} \quad M_\theta = M_0^{1/2} (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2},$$

leading to the Hilbert space $X_\theta = (\mathbb{R}^n, (\cdot,\cdot)_{X_\theta})$. We also write $[X_0, X_1]_\theta$ and $[M_0, M_1]_\theta$ for X_θ and M_θ, respectively.

A second representation of the norm $\|\cdot\|_\theta$

Let $0<\theta<1$. The following identity

$$\int_0^\infty \frac{t^{-(2\theta+1)}}{1+t^{-2}\lambda_i^{-1}}\,dt = \lambda_i^\theta \int_0^\infty \frac{s^{1-2\theta}}{1+s^2}\,ds = \lambda_i^\theta\, c_\theta \quad \text{with } c_\theta = \frac{\pi}{2\sin(\theta\pi)},$$

which follows directly from the substitution rule applied to the transformation $s=f(t)=\sqrt{\lambda_i}t$, allows a different representation of the norm in $[X_0,X_1]_\theta$ (K-method):

$$\begin{aligned}\|x\|_{X_\theta}^2 &= \sum_{i=1}^n \lambda_i^\theta\,\hat{x}_i^2 = c_\theta^{-1}\sum_{i=1}^n \int_0^\infty \frac{t^{-(2\theta+1)}}{1+t^{-2}\lambda_i^{-1}}\,dt\,\hat{x}_i^2 \\ &= c_\theta^{-1}\int_0^\infty t^{-(2\theta+1)}\sum_{i=1}^n \frac{1}{1+t^{-2}\lambda_i^{-1}}\,\hat{x}_i^2\,dt.\end{aligned}$$

Now

$$\begin{aligned}\sum_{i=1}^n \frac{1}{1+t^{-2}\lambda_i^{-1}}\,\hat{x}_i^2 &= \sum_{i=1}^n \frac{t^2\lambda_i}{1+t^2\lambda_i}\,\hat{x}_i^2 = \left\langle (M_0^{-1}+t^{-2}M_1^{-1})^{-1}x, x\right\rangle \\ &= \inf_{x=x_0+x_1}\left(\|x_0\|_{X_0}^2 + t^2\,\|x_1\|_{X_1}^2\right).\end{aligned}$$

Therefore,

$$\|x\|_{X_\theta}^2 = c_\theta^{-1}\int_0^\infty t^{-(2\theta+1)}\,K(t;x)^2\,dt$$

with

$$K(t;x) = \inf_{x=x_0+x_1}\left(\|x_0\|_{X_0}^2 + t^2\,\|x_1\|_{X_1}^2\right)^{1/2}.$$

From this representation it follows that

$$[X_0,X_1]_\theta = [X_1,X_0]_{1-\theta}.$$

by using the substitution rule for $t=1/s$.

The interpolation theorem (cont.)

Theorem 1.2. *Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with*

$$\|Tx\|_{Y_0} \le c_0\,\|x\|_{X_0} \quad \textit{and} \quad \|Tx\|_{Y_1} \le c_1\,\|x\|_{X_1},$$

where the norms $\|\cdot\|_{X_i}$ and $\|\cdot\|_{Y_i}$ are the norms associated to the inner products :

$$(x,y)_{X_i} = \langle M_i x, y\rangle \quad \textit{and} \quad (x,y)_{Y_i} = \langle N_i x, y\rangle \quad \textit{for } i=0,1.$$

Then, for $X_\theta=[X_0,X_1]_\theta$ and $Y_\theta=[Y_0,Y_1]_\theta$, we have

$$\|Tx\|_{Y_\theta} \le c_1^{1-\theta}c_2^\theta\,\|x\|_{X_\theta}.$$

Proof.

$$
\begin{aligned}
\|Tx\|_{Y_\theta}^2 &= c_\theta^{-1} \int_0^\infty t^{-2\theta-1} \inf_{Tx=y_0+y_1} \left(\|y_0\|_{Y_0}^2 + t^2 \|y_1\|_{Y_1}^2\right) \, dt \\
&\le c_\theta^{-1} \int_0^\infty t^{-2\theta-1} \inf_{x=x_0+x_1} \left(\|Tx_0\|_{Y_0}^2 + t^2 \|Tx_1\|_{Y_1}^2\right) \, dt \\
&\le c_\theta^{-1} \int_0^\infty t^{-2\theta-1} \inf_{x=x_0+x_1} \left(c_1^2 \|x_0\|_{X_0}^2 + c_2^2 \, t^2 \|x_1\|_{X_1}^2\right) \, dt \\
&= c_1^2 \, c_\theta^{-1} \int_0^\infty t^{-2\theta-1} \inf_{x=x_0+x_1} \left(\|x_0\|_{X_0}^2 + \left(\frac{c_2}{c_1}\right)^2 t^2 \|x_1\|_{X_1}^2\right) \, dt \\
&= c_1^2 \left(\frac{c_2}{c_1}\right)^{2\theta} c_\theta^{-1} \int_0^\infty s^{-2\theta-1} \inf_{x=x_0+x_1} \left(\|x_0\|_{X_0}^2 + s^2 \|x_1\|_{X_1}^2\right) \, ds \\
&= c_1^{2(1-\theta)} c_2^{2\theta} \|x\|_{X_\theta}^2 .
\end{aligned}
$$

□