Real Interpolation of Sobolev Spaces

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Outline

- **Definition 1:** Sobolev Spaces for $k \in \mathbb{N}_0$

$$W_p^k(\Omega) := \{u \in L^p(\Omega) : \|u\|_{W_p^k(\Omega)} < \infty\}$$

where $D^\alpha u$ is the weak partial derivative and,

$$\|u\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}$$

- Interpolation spaces provide a concept of fractional-order derivatives, extending the definition of Sobolev spaces.

- We choose one definition of fractional order Sobolev spaces;
- **Definition 2:** Sobolev-Slobodečki Spaces
  $(s \in (0,1)\,,\ k \in \mathbb{N}_0)$

$$W_p^{k+s}(\Omega) := \{u \in W_p^k(\Omega) : \|u\|_{W_p^{k+s}(\Omega)} < \infty\}\,,$$

  where

$$\|u\|^p_{W_p^{k+s}(\Omega)} := \|u\|^p_{W_p^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+sp}} \mathrm{d}x\,\mathrm{d}y\,.$$

- We are concerned with $p = 2$ so that $W_p^k(\Omega) = H^k(\Omega)$. For the case $k = 0$, then $H^0(\Omega) = L_2(\Omega)$.

- **Definition 3:** $E : W_p^s(\Omega) \to W_p^s(\mathbb{R}^n)$ is an *extension operator* if E is bounded and satisfies,

$$Eu|_{\Omega} = u \text{ for } u \in W_p^s(\Omega).$$

- In the proof of the theorem we will assume equivalence of norms of the interpolated spaces, a result which will be proved later on.

Theorem

Let $0 < s < 1$. If Ω has a Lipschitz boundary, then

$$W_p^{k+s}(\Omega) = [W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}$$

and the norms are equivalent.

Proof. We assume that the proposition is valid for $\Omega = \mathbb{R}^n$. That is:

$$W_p^{k+s}(\mathbb{R}^n) = [W_p^k(\mathbb{R}^n), W_p^{k+1}(\mathbb{R}^n)]_{s,p}$$

If Ω is Lipschitz, $1 \leq p \leq \infty$, then we have an extension operator $E_S : W_p^k(\Omega) \to W_p^k(\mathbb{R}^n)$ independent of $k \geq 0$.
Interpolating this operator we have,

$$\begin{aligned} \|u\|_{W_p^{k+s}(\Omega)} &= \|E_S u\|_{W_p^{k+s}(\Omega)} \\ &\leq \|E_S u\|_{W_p^{k+s}(\mathbb{R}^n)}, \quad \text{Sobolev-Slobodečki spaces} \\ &\leq C\|E_S u\|_{[W_p^k(\mathbb{R}^n), W_p^{k+1}(\mathbb{R}^n)]_{s,p}}, \quad \text{equivalence of norms} \\ &\leq C\|u\|_{[W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}}, \quad \text{by exact interpolation} \end{aligned}$$

NB: E_s is defined for both $W_p^k(\Omega)$ and $W_p^{k+1}(\Omega)$.

Conversely, there exists an extension operator, $E_G : W_p^{k+s}(\Omega) \to W_p^{k+s}(\mathbb{R}^n)$ such that $E_G u|_\Omega = u$, Ω a Lipschitz domain. Then,

$$\begin{aligned} \|u\|_{[W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}} &= \|E_G u\|_{[W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}} \\ &\leq \|E_G u\|_{[W_p^k(\mathbb{R}^n), W_p^{k+1}(\mathbb{R}^n)]_{s,p}} \\ &\leq C\|E_G u\|_{W_p^{k+s}(\mathbb{R}^n)}, \quad \text{equivalence of norms} \\ &\leq C\|u\|_{W_p^{k+s}(\Omega)}. \end{aligned}$$

As such, the Sobolev-Slobodečki norm is equivalent to the real interpolation norm when Ω is Lipschitz.

□

Theorem

$$H^s(\mathbb{R}^n) \cong [H^0(\mathbb{R}^n), H^1(\mathbb{R}^n)]_s\,, \ \forall s \in (0, 1)\,. \tag{1}$$

Proof.

1. $$H^s(\mathbb{R}^n) \cong H^s_F(\mathbb{R}^n)\,.$$
2. $$H^s_F(\mathbb{R}^n) \cong [H^0_F(\mathbb{R}^n), H^1_F(\mathbb{R}^n)]_s \cong [H^0(\mathbb{R}^n), H^1(\mathbb{R}^n)]_s\,.$$

□

Fourier transformation

Definition (Fourier transformation)

Let $u \in L^1(\mathbb{R}^n)$.

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} u(x)\,\mathrm{d}x\,, \ \forall \xi \in \mathbb{R}^n \tag{2}$$

Properties:

- Isomorphism in $L^2(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |u(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi\,.$$

- $$\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)\,.$$

- $$\hat{u}(x - x_0)(\xi) = \mathrm{e}^{-ix_0\cdot\xi} \hat{u}(x)(\xi)\,.$$

Space $H^s_F(\mathbb{R}^n)$

Definition

$$H^s_F(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \ \|u\|_{H^s_F(\mathbb{R}^n)} < \infty \right\} ,$$

where

$$\|u\|_{H^s_F(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}} . \qquad (3)$$

Theorem

$$H^s_F(\mathbb{R}^n) \cong H^s(\mathbb{R}^n) , \ \forall s \in [0, 1] .$$

Lemma

$$\int_0^\infty t^{-2s-1} \int_{|\omega|=1} \left| \mathrm{e}^{i \frac{t\xi}{|\xi|} \cdot \omega} - 1 \right|^2 \mathrm{d}\omega \, \mathrm{d}t = a_s .$$

Theorem

$$H^s(\mathbb{R}^n) \cong [H^0(\mathbb{R}^n), H^1(\mathbb{R}^n)]_s\,, \ \forall s \in (0,\, 1)\,.$$

Lemma

For fixed real number $A_0, A_1 > 0$, and for a complex number z is

$$\min_{z=z_0+z_1} (A_0 \left|z_0\right|^2 + A_1 \left|z_1\right|^2) = \frac{A_0 A_1}{A_0 + A_1} \left|z\right|^2\,,$$

and that minimum is achieved when $A_0 z_0 = A_1 z_1 = \frac{A_0 A_1}{A_0 + A_1} z\,.$

Lemma

$$\int_0^\infty \frac{t^{1-2s}}{1+t^2}\,\mathrm{d}t = \frac{\pi}{2\sin(\pi s)}\,,\ s \in (0,\, 1)\,.$$

Thank you for listening