# Interpolation in Banach Spaces - The K-Method

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## 1.1 Intermediate Spaces and Interpolation

### 1.1.1 Intermediate Spaces

We are concerned with the construction of Banach spaces $X$ intermediate between two Banach spaces $X_0$ and $X_1$, each embedded in a Hausdorff vector space $\mathcal{X}$. Let $\|\cdot\|_{X_i}$ denote the norm in $X_i$, $i = 0, 1$. In addition, $X_0 \cap X_1$ and $X_0 + X_1 = \{u = u_0 + u_1 : u_0 \in X_0, u_1 \in X_1\}$ are Banach spaces with respect to the norms

$$\begin{aligned} \|u\|_{X_0 \cap X_1} &= \max\{\|u\|_{X_0}, \|u\|_{X_1}\} \\ \|u\|_{X_0 + X_1} &= \inf_{u=u_0+u_1} \{\|u_0\|_{X_0} + \|u_1\|_{X_1}\} \end{aligned}$$

and $X_0 \cap X_1 \to X_i \to X_0 + X_1$, for $i = 0, 1$.
Below, we show only the proof of the triangle inequality of the $\|\cdot\|_{X_0+X_1}$ norm and that the space $X_0 + X_1$ is indeed complete.

*Proof.* Triangle inequality

$$\begin{aligned} \|u + v\|_{X_0+X_1} &= \inf_{u+v=w_0+w_1} \{\|w_0\|_{X_0} + \|w_1\|_{X_1}\} \\ &= \inf_{\substack{u=u_0+u_1 \\ v=v_0+v_1}} \{\|u_0 + v_0\|_{X_0} + \|u_1 + v_1\|_{X_1}\} \\ &\leq \inf\{\|u_0 + v_0\|_{X_0} + \|u_1 + v_1\|_{X_1}\} \\ &\leq \inf\{\|u_0\|_{X_0} + \|v_0\|_{X_0} + \|u_1\|_{X_1} + \|v_1\|_{X_1}\} \\ &= \inf\{\|u_0\|_{X_0} + \|u_1\|_{X_1}\} + \inf\{\|v_0\|_{X_0} + \|v_1\|_{X_1}\}, \quad \text{since the} \\ &\quad \text{infimum is taken over two different spaces} \\ &= \|u\|_{X_0+X_1} + \|v\|_{X_0+X_1} \end{aligned}$$

□

*Proof.* Completeness, c.f [2].
We first assume that $\{u_n\}_{n\in\mathbb{N}} \subset X_0 + X_1$ Cauchy and,

$$\sum_{n=1}^{\infty} \|u_n\|_{X_0+X_1} < \infty$$

Then, considering the decompositions $u_n = u_n^0 + u_n^1$ such that,

$$\|u_n^0\|_{X_0} + \|u_n^1\|_{X_1} \leq 2\|u_n\|_{X_0+X_1}$$

We see then that,

$$\sum_{n=1}^{\infty} \|u_n^0\|_{X_0} < \infty, \quad \sum_{n=1}^{\infty} \|u_n^1\|_{X_1} < \infty$$

We use the lemma,

**Lemma 1.** Suppose that $X$ is a normed vector space. Then $X$ is complete if and only if

$$\sum_{n=1}^{\infty} \|u_n\|_X < \infty,$$

implies that there is an element $u \in X$ such that

$$\|u - \sum_{n=1}^{N} u_n\|_X \to 0, \quad N \to \infty$$

Since $X_i$, $i = 0, 1$ are complete, then by lemma 1, we know that $\sum u_n^i$ converges in $X_i$, $i = 0, 1$ respectively. If we let $u^0 = \sum u_n^0$ and $u^1 = \sum u_n^1$ and $u = u^0 + u^1$, then

$$\begin{aligned} \|u - \sum_{n=1}^{N} u_n\|_{X_0+X_1} &= \|u^0 + u^1 - \sum_{n=1}^{N}(u_n^0 + u_n^1)\|_{X_0+X_1} \\ &\leq \|u^0 - \sum_{n=1}^{N} u_n^0\|_{X_0} + \|u^1 - \sum_{n=1}^{N} u_n^1\|_{X_1} \to 0, \quad (N \to \infty) \end{aligned}$$

Then, $\sum u_n$ converges in $X_0 + X_1$ to $u \Rightarrow X_0 + X_1$ complete. □

The Banach space $X_i$ is *intermediate* if the embeddings above exist.

### 1.1.2 The K norm

For each fixed $t > 0$, the functional *K(t;u)* defines a norm in $X_0 + X_1$ as,

$$K(t; u) = \inf\{\|u_0\|_{X_0} + t\|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}$$

equivalent to $\|\cdot\|_{X_0+X_1}$. We also have that,

$$\min\{1, t\}\|u\|_{X_0+X_1} \leq K(t; u) \leq \max\{1, t\}\|u\|_{X_0+X_1} \tag{1.1}$$

For $0 < a < b$ and $0 < \theta < 1$, we have,

$$\begin{aligned} &\|u_0\|_{X_0} + ((1-\theta)a + \theta b)\|u_1\|_{X_1} \\ &\quad = (1-\theta)(\|u_0\|_{X_0} + a\|u_1\|_{X_1}) + \theta(\|u_0\|_{X_0} + b\|u_1\|_{X_1}) \\ &\quad \geq (1-\theta)K(a; u) + \theta K(b; u) \end{aligned}$$

so that $K(t; u)$ is a concave function of $t$.

If $u \in X_0 \cap X_1$, then for any $t > 0$ we have $K(t; u) \leq \|u\|_{X_0}$ and $K(t; u) \leq t\|u\|_{X_1}$.

### 1.1.3 The K-method

If $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$, let $(X_0, X_1)_{\theta,q;K}$ denote the space of all $u \in X_0+X_1$ such that the function $t \to t^{-\theta}K(t; u)$ belongs to $L^q_* = L^q\left(0, \infty; \frac{dt}{t}\right)$.

**Theorem 2.** If and only if either $1 \leq q < \infty$ and $0 < \theta < 1$ or $q = \infty$ and $0 \leq \theta \leq 1$, then the space $(X_0, X_1)_{\theta,q;K}$ is a non-trivial Banach space with norm

$$\|u\|_{\theta,q;K} = \begin{cases} \left(\int_0^\infty \left(t^{-\theta}K(t; u)\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \leq q < \infty \\ \operatorname{ess\,sup}_{0<t<\infty}\{t^{-\theta}K(t; u)\} & \text{if } q = \infty \end{cases}$$

Furthermore,

$$\|u\|_{X_0+X_1} \leq \frac{\|u\|_{\theta,q;K}}{\|t^{-\theta}\min\{1, t\}\|_{L^q_*}} \leq \|u\|_{X_0\cap X_1} \tag{1.2}$$

So the embeddings,

$$X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;K} \to X_0 + X_1$$

hold, and $(X_0, X_1)_{\theta,q;K}$ is an intermediate space between $X_0$ and $X_1$. Otherwise $(X_0, X_1)_{\theta,q;K}= \{0\}$, (i.e $\theta = 0$ or $\theta = 1$).

*Proof.* With the usage of the fact that:

$$\int_0^1 \frac{1}{t^\alpha} < \infty \Leftrightarrow \alpha < 1 \qquad\qquad \int_1^\infty \frac{1}{t^\alpha} < \infty \Leftrightarrow \alpha > 1$$

we can easily prove that function $t \mapsto t^{-\theta} \min\{1,\, t\}$ belongs to $L^q_*$ if and only if $\theta$ and $q$ satisfy the conditions of the theorem. Since (1.1) we have:

$$\|t^{-\theta} \min\{1,\, t\}\|_{L^q_*} \|u\|_{X_0+X_1} \leq \|t^{-\theta} K(t;u)\|_{L^q_*} = \|u\|_{\theta,q;K} \tag{1.3}$$

this means that if $\exists\ u \neq 0 \in (X_0, X_1)_{\theta,q;K}$ the conditions for $\theta$ and $q$ must be satisfied.
On the other hand we have following estimates:

$$K(t;u) \leq \|u\|_{X_0} \leq \max\{\|u\|_{X_0}, \|u\|_{X_1}\}$$

$$K(t;u) \leq t\|u\|_{X_1} \leq t \max\{\|u\|_{X_0}, \|u\|_{X_1}\}$$

From whose immediately:

$$K(t;u) \leq \min\{1,\, t\} \max\{\|u\|_{X_0}, \|u\|_{X_1}\} = \min\{1,\, t\}\|u\|_{X_0 \cap X_1}$$

So, we get following estimate:

$$\|u\|_{\theta,q;K} = \|t^{-\theta} K(t;u)\|_{L^q_*} \leq \|t^{-\theta} \min\{1,\, t\}\|_{L^q_*} \|u\|_{X_0 \cap X_1} \tag{1.4}$$

This means that if the conditions for $\theta$ and $q$ are satisfied every element of $X_0 \cap X_1$ belongs also to the space $(X_0, X_1)_{\theta,q;K}$. Because we assume that the intersection $X_0 \cap X_1$ is non-trivial, and also the space $(X_0, X_1)_{\theta,q;K}$ is non-trivial when the conditions are satisfied.
Moreover estimates (1.3) and (1.4) show that embeddings (1.2) hold.
Now, we prove that $\|\cdot\|_{\theta,q;K}$ is a norm. Because it is easy to see that $\|u\|_{\theta,q;K} \geq 0$ and $\|\alpha u\|_{\theta,q;K} = |\alpha|\, \|u\|_{\theta,q;K}$ we only have to prove that the triangle inequality is satisfied.

- For the case $1 \leq q < \infty$ we use that $K(t; u+v) \leq K(t;u) + K(t;v)$ and Minkowski inequality to get:

$$\begin{aligned}
\|u+v\|_{\theta,q;K} &= \left( \int_0^\infty (t^{-\theta} K(t; u+v))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\leq \left( \int_0^\infty (t^{-\theta} t^{-\frac{1}{q}} K(t;u))^q dt + \int_0^\infty ((t^{-\theta} t^{-\frac{1}{q}} K(t;v))^q dt \right)^{\frac{1}{q}} \\
&\leq \left( \int_0^\infty \left(t^{-\theta} K(t;u)\right)^q \frac{dt}{t} \right)^{1/q} + \left( \int_0^\infty \left(t^{-\theta} K(t;v)\right)^q \frac{dt}{t} \right)^{1/q} \\
&= \|u\|_{\theta,q;K} + \|v\|_{\theta,q;K} \,.
\end{aligned}$$

- For $q = \infty$:

$$\text{ess sup}_{0<t<\infty}\{t^{-\theta} K(t; u+v)\} \leq \text{ess sup}_{0<t<\infty}\{t^{-\theta} K(t;u)\} + \text{ess sup}_{0<t<\infty}\{t^{-\theta} K(t;v)\} \,.$$

□

According to [1] the space $(X_0, X_1)_{\theta,q;K}$ is complete under the norm $\|\cdot\|_{\theta,q;K}$.

## 1.2 Interpolation theorems

**Theorem 3** (Interpolation theorem I)**.** Let $T : X_0 + X_1 \mapsto Y_0 + Y_1$ be a linear operator with properties

$$\|Tx\|_{Y_0} \le c_0 \|x\|_{X_0} \qquad \text{and} \qquad \|Tx\|_{Y_1} \le c_1 \|x\|_{X_1} \,.$$

Then

$$\|Tx\|_{Y_0+Y_1} \le \max\{c_0,\, c_1\} \|x\|_{X_0+X_1} \tag{1.5}$$

and

$$\|Tx\|_{Y_0\cap Y_1} \le \max\{c_0,\, c_1\} \|x\|_{X_0\cap X_1} \,. \tag{1.6}$$

*Proof.*

1.

$$\begin{aligned}
\|Tx\|_{Y_0+Y_1} &= \inf_{Tx=y_0+y_1} \{\|y_0\|_{Y_0} + \|y_1\|_{Y_1}\} \\
&\le \inf_{x=x_0+x_1} \{\|Tx_0\|_{Y_0} + \|Tx_1\|_{Y_1}\} \\
&\le \inf_{x=x_0+x_1} \{c_0\|x_0\|_{X_0} + c_1\|x_1\|_{X_1}\} \\
&\le \max\{c_0, c_1\} \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + \|x_1\|_{X_1}\} \\
&= \max\{c_0,\, c_1\} \|x\|_{X_0+X_1} \,.
\end{aligned}$$

2.

$$\begin{aligned}
\|Tx\|_{Y_0\cap Y_1} &= \max\{\|Tx\|_{Y_0}, \|Tx\|_{Y_1}\} \\
&\le \max\{c_0\|x\|_{X_0}, c_1\|x\|_{X_1}\} \\
&\le \max\{c_0, c_1\} \max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\} \\
&= \max\{c_0,\, c_1\} \|x\|_{X_0\cap X_1} \,.
\end{aligned}$$

□

**Theorem 4** (An Exact Interpolation Theorem)**.** Let $T : X_0 + X_1 \mapsto Y_0 + Y_1$ be a linear operator with properties

$$\|Tx\|_{Y_0} \le c_0 \|x\|_{X_0} \qquad \text{and} \qquad \|Tx\|_{Y_1} \le c_1 \|x\|_{X_1} \,,$$

let either $0 < \theta < 1$ and $1 \le q < \infty$ or $0 \le \theta \le 1$ and $q = \infty$. Then for $X_\theta = (X_0,\, X_1)_{\theta,q;K}$ and $Y_\theta = (Y_0,\, Y_1)_{\theta,q;K}$

$$\|Tx\|_{Y_\theta} \le c_0^{1-\theta} c_1^{\theta} \|x\|_{X_\theta} \,. \tag{1.7}$$

*Proof.*

- $1 \leq q < \infty$

$$\begin{aligned}
\|T_x\|^q_{Y_\theta} &= \int_0^\infty \left( t^{-\theta} \inf_{Tx=y_0+y_1} \{\|y_0\|_{Y_0} + t\|y_1\|_{Y_1}\} \right)^q \frac{dt}{t} \\
&\leq \int_0^\infty \left( t^{-\theta} \inf_{x=x_0+x_1} \{\|Tx_0\|_{Y_0} + t\|Tx_1\|_{Y_1}\} \right)^q \frac{dt}{t} \\
&\leq \int_0^\infty \left( t^{-\theta} \inf_{x=x_0+x_1} \{c_0\|x_0\|_{X_0} + c_1 t\|x_1\|_{X_1}\} \right)^q \frac{dt}{t} \\
&= c_0^q \int_0^\infty \left( t^{-\theta} \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + \frac{c_1}{c_0} t\|x_1\|_{X_1}\} \right)^q \frac{dt}{t} \\
&= c_0^q \left(\frac{c_0}{c_1}\right)^{-\theta q} \int_0^\infty \left( s^{-\theta} \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + s\|x_1\|_{X_1}\} \right)^q \frac{ds}{s} \\
&= c_0^{(1-\theta)q} c_1^{\theta q} \|x\|^q_{X_\theta} .
\end{aligned}$$

- $q = \infty$

$$\begin{aligned}
\|Tx\|_{Y_\theta} &= \text{ess sup}_{0<t<\infty} \left\{ t^{-\theta} \inf_{Tx=y_0+y_1} \{\|y_0\|_{Y_0} + t\|y_1\|_{Y_1}\} \right\} \\
&\leq \text{ess sup}_{0<t<\infty} \left\{ t^{-\theta} \inf_{x=x_0+x_1} \{\|Tx_0\|_{Y_0} + t\|Tx_1\|_{Y_1}\} \right\} \\
&\leq \text{ess sup}_{0<t<\infty} \left\{ t^{-\theta} \inf_{x=x_0+x_1} \{c_0\|x_0\|_{X_0} + c_1 t\|x_1\|_{X_1}\} \right\} \\
&=\leq \text{ess sup}_{0<t<\infty} \left\{ t^{-\theta} c_0 \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + \frac{c_1}{c_0} t\|x_1\|_{X_1}\} \right\} \\
&= c_0 \left(\frac{c_1}{c_0}\right)^\theta \text{ess sup}_{0<t<\infty} \left\{ t^{-\theta} \left(\frac{c_1}{c_0}\right)^{-\theta} \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + \frac{c_1}{c_0} t\|x_1\|_{X_1}\} \right\} \\
&= c_0^{1-\theta} c_1^\theta \text{ess sup}_{0<s<\infty} \left\{ s^{-\theta} \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + s\|x_1\|_{X_1}\} \right\} \\
&= c_0^{1-\theta} c_1^\theta \|x\|_{X_\theta} .
\end{aligned}$$

□

**Definition 5.** Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be two interpolation pairs of Banach spaces. Let $X$ and $Y$ be intermediate spaces of $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ respectively. We say that $X$ and $Y$ are *interpolation spaces of type* $\theta$, $0 \leq \theta \leq 1$, if every bounded operator $T : X_0 + X_1 \mapsto Y_0 + Y_1$, $T$ bounded from $X_i$ into $Y_i$ with norm at most $c_i$, maps $X$ into $Y$ with norm $c$ satisfying

$$c \leq K c_0^{(1-\theta)} c_1^\theta , \tag{1.8}$$

where constant $K \leq 1$ independent of $T$. We say that the interpolation spaces $X$ and $Y$ are *exact* if inequality (1.8) holds with $K = 1$.

*Remark* 6. According to theorem 4 the spaces $X_\theta$ and $Y_\theta$ are exact.

# Bibliography


[1] Adams, R.A., Fournier J.J.F *Sobolev spaces.* Academic Press, 2003.

[2] Bergh J.,Lofstrom J. *Interpolation Spaces - An Introduction.* Springer, 1976.