Real Interpolation of Sobolev Spaces

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1.1 Real Interpolation of Sobolev Spaces

1.1.1 Definitions

Definition 1. Sobolev spaces for $k \in \mathbb{N}_0$ are defined as follows,

$$W_p^k(\Omega) := \{u \in L^p(\Omega) : \|u\|_{W_p^k(\Omega)} < \infty\}$$

where $D^\alpha u$ is the weak partial derivative.

$$\|u\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}$$

denotes the standard norm in $W_p^k(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Interpolation spaces provide a concept of fractional-order derivatives, extending the definition of Sobolev spaces. We choose one definition of interpolation spaces to motivate the following definition of fractional order Sobolev spaces;

Definition 2. Sobolev-Slobodečki Spaces $(s \in (0,1)\,,\ k \in \mathbb{N}_0)$

$$W_p^{k+s}(\Omega) := \{u \in W_p^k(\Omega) : \|u\|_{W_p^{k+s}(\Omega)} < \infty\}\,,$$

where

$$\|u\|_{W_p^{k+s}(\Omega)}^p := \|u\|_{W_p^k(\Omega)}^p + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+sp}} \mathrm{d}x\mathrm{d}y\,.$$

We are concerned with the case $p = 2$ and will use interchangeably $W_p^k(\Omega)$ and $H^k(\Omega)$ i.e $W_p^k(\Omega) = H^k(\Omega)$. For the case $k = 0$, then $H^0(\Omega) = L_2(\Omega)$.

Definition 3. $E : W_p^s(\Omega) \to W_p^s(\mathbb{R}^n)$ is an *extension operator* if E is linear, bounded and satisfies the condition,

$$Eu|_\Omega = u \quad \text{for } u \in W_p^s(\Omega). \quad [2]$$

In the proof of the following theorem we will make use of the equivalence of norms of interpolated spaces for $\Omega = \mathbb{R}^n$, a result that will be proved later on.

Theorem 4. Let $0 < s < 1$. If Ω has a Lipschitz boundary, then

$$W_p^{k+s}(\Omega) = [W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}$$

and the norms are equivalent.

Proof. We assume that the proposition is valid for $\Omega = \mathbb{R}^n$. That is:

$$W_p^{k+s}(\mathbb{R}^n) = [W_p^k(\mathbb{R}^n), W_p^{k+1}(\mathbb{R}^n)]_{s,p}$$

If Ω is Lipschitz, $1 \le p \le \infty$, then we have an extension operator $E_k : W_p^k(\Omega) \to W_p^k(\mathbb{R}^n)$ defined for $W_p^k(\Omega)$ and is also an extension operator on $W_p^{k+1}(\Omega)$ [2]. Interpolating this operator we have,

$$\begin{aligned}
\|u\|_{W_p^{k+s}(\Omega)} &= \|E_k u\|_{W_p^{k+s}(\Omega)} \\
&\le \|E_k u\|_{W_p^{k+s}(\mathbb{R}^n)}, \quad \text{Sobolev-Slobodečki spaces} \\
&\le C\|E_k u\|_{[W_p^k(\mathbb{R}^n), W_p^{k+1}(\mathbb{R}^n)]_{s,p}}, \quad \text{by equivalence of norms} \\
&\le C\|u\|_{[W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}}, \quad \text{by exact interpolation}
\end{aligned}$$

Conversely, there exists an extension operator, $E_G : W_p^{k+s}(\Omega) \to W_p^{k+s}(\mathbb{R}^n)$ such that $E_G u|_\Omega = u$, Ω a Lipschitz domain.
We then have,

$$\begin{aligned}
\|u\|_{[W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}} &= \|E_G u\|_{[W_p^k(\Omega), W_p^{k+1}(\Omega)]_{s,p}} \\
&\le \|E_G u\|_{[W_p^k(\mathbb{R}^n), W_p^{k+1}(\mathbb{R}^n)]_{s,p}} \\
&\le C\|E_G u\|_{W_p^{k+s}(\mathbb{R}^n)}, \quad \text{by equivalence of norms} \\
&\le C\|u\|_{W_p^{k+s}(\Omega)}.
\end{aligned}$$

As such, the Sobolev-Slobodečki norm is equivalent to the real interpolation norm when Ω is Lipschitz.

□

1.2 Equivalence for $\Omega = \mathbb{R}^n$

In this section we would like to prove[1]

$$W_p^{k+s}(\mathbb{R}^n) \cong [W_p^k(\mathbb{R}^n), W_p^{k+1}(\mathbb{R}^n)]_{s,p}\,, \tag{1.1}$$

which was our assumption in the proof of Theorem 4.

For simplicity we will prove, that (1.1) is valid for $p = 2$ and $k = 0$, i.e.

$$H^s(\mathbb{R}^n) \cong [H^0(\mathbb{R}^n)\,, H^1(\mathbb{R}^n)]_s\,. \tag{1.2}$$

To prove (1.2) we will introduce a new space $H_F^s(\mathbb{R}^n)$ and show that:

$$H^s(\mathbb{R}^n) \cong H_F^s(\mathbb{R}^n) \cong [H_F^0(\mathbb{R}^n)\,, H_F^1(\mathbb{R}^n)]_s = [H^0(\mathbb{R}^n)\,, H^1(\mathbb{R}^n)]_s\,.$$

1.2.1 The Space $H_F^s(\mathbb{R}^n)$

In this subsection we will introduce the new space $H_F^s(\mathbb{R}^n)$. To do this we recall the definition of the Fourier transform in $\mathbb{R}^n$ and its basic properties.

Definition 5 (Fourier transformation)**.** Let $u \in L^2(\mathbb{R}^n)$. The Fourier-transform $\hat{u}$ of u is defined by the formula:

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} u(x)\,\mathrm{d}x\,,\ \forall \xi \in \mathbb{R}^n\,. \tag{1.3}$$

The Fourier transformation has the following properties [3]:

- Isomorphism in $L^2(\mathbb{R}^n)$:
$$\int_{\mathbb{R}^n} |u(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2\,\mathrm{d}\xi\,.$$
- $$\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)\,.$$
- $$\hat{u}(x - x_0)(\xi) = \mathrm{e}^{-ix_0\cdot\xi}\hat{u}(x)(\xi)\,.$$

Now we are in the position to define the new space $H_F^s(\mathbb{R}^n)$.

Definition 6.

$$H_F^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) :\ \|u\|_{H_F^s(\mathbb{R}^n)} < \infty \right\}\,,$$

where

$$\|u\|_{H_F^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2\,\mathrm{d}\xi \right)^{\frac{1}{2}}\,. \tag{1.4}$$

[1] We introduced the new symbol $\cong$ which means that the sets are the same and the norms are equivalent.

To show that $H_F^s(\mathbb{R}^n)$ is equivalent to $H^s(\mathbb{R}^n)$ we will use the following lemma and properties of the Fourier transform mentioned above.

Lemma 7. Let $0 < s < 1$. The integral

$$a_s := \int_0^\infty t^{-2s-1} \int_{|\omega|=1} \left| \mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1 \right|^2 \mathrm{d}\omega\, \mathrm{d}t\,, \tag{1.5}$$

does not depend on ξ and is finite, that is $0 < a_s < \infty$.

Proof. In order to prove this lemma, we have to introduce spherical coordinates. For simplicity, we consider the two-dimensional case $n = 2$. We split the integral into two integrals:

$$\begin{aligned} &\int_0^\infty t^{-2s-1} \int_{|\omega|=1} \left| \mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1 \right|^2 \mathrm{d}\omega\, \mathrm{d}t \\ &= \int_0^1 t^{-2s-1} \int_{|\omega|=1} \left| \mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1 \right|^2 \mathrm{d}\omega\, \mathrm{d}t + \int_1^\infty t^{-2s-1} \int_{|\omega|=1} \left| \mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1 \right|^2 \mathrm{d}\omega\, \mathrm{d}t\,. \end{aligned} \tag{1.6}$$

The second integral is finite because:

$$\left| \mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1 \right|^2 \leq 4$$

and therefore:

$$\int_1^\infty t^{-2s-1} \int_{|\omega|=1} \left| \mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1 \right|^2 \mathrm{d}\omega\, \mathrm{d}t \leq \int_1^\infty t^{-2s-1} 8\pi\, \mathrm{d}t \ < \infty\,,$$

for $s > 0$.

Let consider the first integral from 0 to 1. The inner integral can be written as:

$$\begin{aligned} \int_{|\omega|=1} \left| \mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1 \right|^2 \mathrm{d}\omega &= \int_{|\omega|=1} \left| \cos\left(\frac{t\xi}{|\xi|}\cdot\omega\right) - 1 + i\sin\left(\frac{t\xi}{|\xi|}\cdot\omega\right) \right|^2 \mathrm{d}\omega = \\ &= \int_{|\omega|=1} 2 - 2\cos\left(\frac{t\xi}{|\xi|}\cdot\omega\right) \mathrm{d}\omega \end{aligned} \tag{1.7}$$

Now, we will use the estimate

$$\cos x \geq 1 - \frac{x^2}{2}\,, \qquad \text{for } 0 \leq x \leq 1\,. \tag{1.8}$$

We can do this because the argument in cosine in our case is smaller than 1:

$$\frac{t\xi}{|\xi|}\cdot\omega \leq \frac{t}{|\xi|}|\xi||\omega| \leq t|\omega| \leq 1\,.$$

After substituting (1.8) into (1.7) we get

$$\int_{|\omega|=1} 2-2\cos\left(\frac{t\xi}{|\xi|}\cdot\omega\right)\mathrm{d}\omega \leq \int_{|\omega|=1} 2-2+\left(\frac{t\xi}{|\xi|}\cdot\omega\right)^2\mathrm{d}\omega = t^2\int_{|\omega|=1}\left(\frac{\xi\cdot\omega}{|\xi|}\right)^2$$

Using polar substitution:

$$t^2\int_{|\omega|=1}\left(\frac{\xi\cdot\omega}{|\xi|}\right)^2 = t^2\int_0^{2\pi}\left(\frac{\xi_1}{|\xi|}\cos\varphi+\frac{\xi_2}{|\xi|}\sin\varphi\right)^2\mathrm{d}\varphi\,.$$

Also we know that there exists some angle θ such that $\frac{\xi_1}{|\xi|}=\cos\theta$ and $\frac{\xi_2}{|\xi|}=\sin\theta$ and therefore:

$$t^2\int_0^{2\pi}\left(\frac{\xi_1}{|\xi|}\cos\varphi+\frac{\xi_2}{|\xi|}\sin\varphi\right)^2\mathrm{d}\varphi = t^2\int_0^{2\pi}(\cos\theta\cos\varphi+\sin\theta\sin\varphi)^2\,\mathrm{d}\varphi = \int_0^{2\pi}\cos^2(\varphi-\theta)\,\mathrm{d}\varphi = t^2\pi\,.$$

We get the estimate for the inner integral:

$$\int_{|\omega|=1}\left|\mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega}-1\right|^2\mathrm{d}\omega \leq t^2\pi\,.$$

Follows that:

$$\int_0^1 t^{-2s-1}\int_{|\omega|=1}\left|\mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega}-1\right|^2\mathrm{d}\omega\,\mathrm{d}t \leq \pi\int_0^1 t^{-2s+1}\,\mathrm{d}t < \infty\,,$$

for $s<1$.
In conclusion the whole integral (1.5) is finite for $0<s<1$.
The independence of ξ follows from rotation invariance of expression in the inner integral. □

Theorem 8.

$$H^s_F(\mathbb{R}^n)\cong H^s(\mathbb{R}^n)\,,\ \forall s\in[0,\,1]\,.$$

Proof. Let us consider the cases $s=0$, $s=1$, and $s\in(0,1)$ separately:

- $s=0$:

$$\|u\|_{H^0_F} = \|\hat{u}\|_{L^2} = \|u\|_{L^2}\,.$$

- $s = 1$:

$$\begin{aligned}
\|u\|^2_{H^1_F} &= \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}^n} \underbrace{|i|^2}_{=1} |\xi|^2 \, |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} |\hat{u}(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} \sum_{j=1}^{n} |i\xi_j|^2 \, |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} |\hat{u}(x)|^2 \,\mathrm{d}x + \sum_{|\alpha|=1} \int_{\mathbb{R}^n} |(i\xi)^\alpha \hat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} |\hat{u}(x)|^2 \,\mathrm{d}x + \sum_{|\alpha|=1} \int_{\mathbb{R}^n} \left|\widehat{D^\alpha u}(\xi)\right|^2 \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} |\hat{u}(x)|^2 \,\mathrm{d}x + \sum_{|\alpha|=1} \int_{\mathbb{R}^n} |D^\alpha u(\xi)|^2 \,\mathrm{d}\xi = \|u\|^2_{H^1} \,.
\end{aligned}$$

- $0 < s < 1$: After substituting $y = x + h$ we get, from Definition 2:

$$\begin{aligned}
\|u\|^2_{H^s} &= \int_{\mathbb{R}^n} |u(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x+h) - u(x)|^2}{|h|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}h \\
&= \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}^n} \frac{1}{|h|^{n+2s}} \int_{\mathbb{R}^n} |u(x+h) - u(x)|^2 \,\mathrm{d}x \,\mathrm{d}h \\
&= \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}^n} \frac{1}{|h|^{n+2s}} \int_{\mathbb{R}^n} |\hat{u}(x+h)(\xi) - \hat{u}(x)(\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}h \\
&= \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \int_{\mathbb{R}^n} \frac{\left|\mathrm{e}^{ih\cdot\xi} - 1\right|^2}{|h|^{n+2s}} \,\mathrm{d}h \,\mathrm{d}\xi \,. \qquad (1.9)
\end{aligned}$$

Now we take the inner integral in the last term of (1.9) and use generalized polar substitution $h = \rho\omega$, $\rho = |h|$, $\omega = \frac{h}{|h|}$, $\mathrm{d}h = \rho^{n-1} \,\mathrm{d}\rho \mathrm{d}\omega$, so that:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\left|\mathrm{e}^{ih\cdot\xi} - 1\right|^2}{|h|^{n+2s}} \,\mathrm{d}h = \int_0^\infty \rho^{-1-2s} \int_{|\omega|=1} |\mathrm{e}^{i\rho\omega\cdot\xi} - 1|^2 \,\mathrm{d}\omega \,\mathrm{d}\xi \\
&= \int_0^\infty t^{-1-2s} |\xi|^{1+2s} \int_{|\omega|=1} \left|\mathrm{e}^{i\frac{t\xi}{|\xi|}\cdot\omega} - 1\right|^2 \,\mathrm{d}\omega \frac{1}{|\xi|} \,\mathrm{d}t = a_s |\xi|^{2s} \,, \qquad (1.10)
\end{aligned}$$

by using the substitution $\rho = t/|\xi|$, $\mathrm{d}\rho = 1/|\xi| \,\mathrm{d}t$ and Lemma 7.

Now after plugging (1.10) into (1.9):

$$\|u\|^2_{H^s} = \int_{\mathbb{R}^n} \left(1 + a_s |\xi|^{2s}\right) |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \approx \|u\|^2_{H^s_F} \,, \qquad (1.11)$$

because there exists constants C_1 and C_2 such that

$$C_1 (1 + |\xi|^2)^s \leq 1 + a_s |\xi|^{2s} \leq C_2 (1 + |\xi|^2)^s \,. \qquad (1.12)$$

These constants exist because both expressions in (1.12) have the same limit in ∞.

□

1.2.2 Equivalence Theorem

In this subsection we will prove the equivalence of the spaces $H^s(\mathbb{R}^n)$ and $[H^0(\mathbb{R}^n), H^1(\mathbb{R}^n)]_s$. In fact, we will prove the equivalence of the spaces $H^s_F(\mathbb{R}^n)$ and $[H^0_F(\mathbb{R}^n), H^1_F(\mathbb{R}^n)]_s$. But according to Theorem 8 we know that the spaces $H^s(\mathbb{R}^n)$ and $H^s_F(\mathbb{R}^n)$ are equivalent for $s \in [0,1]$. From this follows also $[H^0_F(\mathbb{R}^n), H^1_F(\mathbb{R}^n)]_s = [H^0(\mathbb{R}^n), H^1(\mathbb{R}^n)]_s$.

For the proof we will need the following two lemmas.

Lemma 9. For fixed real numbers $A_0, A_1 > 0$, and for a complex number z

$$\min_{z=z_0+z_1} (A_0 |z_0|^2 + A_1 |z_1|^2) = \frac{A_0 A_1}{A_0 + A_1} |z|^2 \,,$$

and that minimum is achieved when $A_0 z_0 = A_1 z_1 = \frac{A_0 A_1}{A_0+A_1} z$.

Proof. Let z_0, z_1 be arbitrary complex numbers such that $z = z_0 + z_1$. Let

$$z_0 = x + iy \,. \tag{1.13}$$

Then

$$z_1 = (\Re z - x) + i(\Im z - y) \,. \tag{1.14}$$

Our task is to find minimum of the function f of two variables x, y:

$$f(x,y) = A_0(x^2 + y^2) + A_1 \left((\Re z - x)^2 + (\Im z - y)^2 \right) \,. \tag{1.15}$$

We compute the partial derivative of f and set it equal to zero:

$$\frac{\partial f}{\partial x} = 2A_0 x - 2A_1(\Re z - x) = 0$$
$$\frac{\partial f}{\partial y} = 2A_0 y - 2A_1(\Im z - y) = 0 \,.$$

This system of linear equations has following solution:

$$x = \frac{A_1}{A_0 A_1} \Re z \,, \quad y = \frac{A_1}{A_0 A_1} \Im z \,. \tag{1.16}$$

In order to be sure that for these values the function is minimum, we compute Hessian matrix of f:

$$\nabla^2 f = \begin{pmatrix} 2A_0 + 2A_1 & 0 \\ 0 & 2A_0 + 2A_1 \end{pmatrix} \,,$$

Given that A_0 and A_1 are positive, this Hessian matrix is positive definite and therefore we can conclude that for the values in (1.16), we have a minimum. To compute the minimal value of f we substitute (1.16) into (1.15):

$$\begin{aligned} f_{min}(x,y) &= A_0 \left(\frac{A_1^2}{(A_0 + A_1)^2} (\Re z)^2 + \frac{A_1^2}{(A_0 + A_1)^2} (\Im z)^2 \right) \\ &\quad + A_1 \left(\frac{A_0^2}{(A_0 + A_1)^2} (\Re z)^2 + \frac{A_0^2}{(A_0 + A_1)^2} (\Im z)^2 \right) \\ &= \frac{A_0 A_1 (A_0 + A_1)}{(A_0 + A_1)^2} ((\Re z)^2 + (\Im z)^2) = \frac{A_0 A_1}{A_0 + A_1} |z|^2 \,. \end{aligned}$$

If we also substitute (1.16) into (1.13) and (1.14) we obtain:

$$z_0 = \frac{A_1}{A_0 + A_1} z\,, \quad z_1 = \frac{A_0}{A_0 + A_1} z\,,$$

i.e. that z_0 and z_1 satisfy the relationships given in the statement of the lemma.

□

Lemma 10. The integral

$$\int_0^\infty \frac{t^{1-2s}}{1+t^2}\, \mathrm{d}t$$

is finite for $s \in (0,\, 1)$.

Proof.

$$\begin{aligned}\int_0^\infty \frac{t^{1-2s}}{1+t^2}\, \mathrm{d}t &= \int_0^1 \frac{t^{1-2s}}{1+t^2}\, \mathrm{d}t + \int_1^\infty \frac{t^{1-2s}}{1+t^2}\, \mathrm{d}t \\ &\leq \int_0^1 t^{1-2s}\, \mathrm{d}t + \int_1^\infty \frac{t^{1-2s}}{t^2}\, \mathrm{d}t = \frac{1}{2-2s} + \frac{1}{2s} < \infty\,,\end{aligned}$$

for $0 < s < 1$. □

Remark 11. Using contour integration we can evaluate the value of the previous integral:

$$\int_0^\infty \frac{t^{1-2s}}{1+t^2}\, \mathrm{d}t = \frac{\pi}{2\sin(\pi s)}\,, \quad s \in (0\,, 1)\,.$$

Theorem 12.

$$H^s(\mathbb{R}^n) \cong [H^0(\mathbb{R}^n), H^1(\mathbb{R}^n)]_s\,,\ \forall s \in (0,\, 1)\,.$$

Proof. According to the discussion at the beginning of this subsection, we will use the Fourier norm $\|\cdot\|_{H^s_F}$. Recall the definition of K-functional:

$$K^2(t;u) = \inf\{\|u_0\|^2_{H^0} + t^2\|u_1\|^2_{H^1} : u = u_0 + u_1, u_0 \in H^0, u_1 \in H^1\}\,.$$

and the norm of the interpolation space

$$\|u\|^2_{K,s} = \int_0^\infty \left|t^{-s}K(t;u)\right|^2 \frac{\mathrm{d}t}{t}\,. \tag{1.17}$$

Let $u = u_0 + u_1$, $u_0 \in H^0$, $u_1 \in H^1$. Then

$$\|u_0\|^2_{H^0} + t^2\|u_1\|^2_{H^1} = \int_{\mathbb{R}^n} \underbrace{1}_{A_0} |\hat{u}_0(\xi)|^2 + \underbrace{t^2(1+|\xi|^2)}_{A_1} |\hat{u}_1(\xi)|^2\, \mathrm{d}\xi\,. \tag{1.18}$$

Since we have $\hat{u}(\xi) = \hat{u}_0(\xi) + \hat{u}_1(\xi)$, according to Lemma 9 the integrand in (1.18) is minimal $\forall \xi$ when u_0 and u_1 are such that:

$$\hat{u}_0(\xi) = t^2(1+|\xi|^2)\hat{u}_1(\xi) = \frac{t^2(1+|\xi|^2)}{1+t^2(1+|\xi|^2)}\hat{u}(\xi)$$

it follows that,

$$K^2(t;u) = \int_{\mathbb{R}^n} \frac{t^2(1+|\xi|^2)}{1+t^2(1+|\xi|^2)} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \,. \tag{1.19}$$

Now we can continue from (1.17):

$$\begin{aligned} \|u\|_{K,s}^2 &= \int_0^\infty \left|t^{-s}K(t;u)\right|^2 \frac{\mathrm{d}t}{t} = \int_0^\infty t^{-2s} \int_{\mathbb{R}^n} \frac{t^2(1+|\xi|^2)}{1+t^2(1+|\xi|^2)} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \frac{\mathrm{d}t}{t} \\ &= \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \int_0^\infty t^{-2s-1} \frac{t^2(1+|\xi|^2)}{1+t^2(1+|\xi|^2)} \,\mathrm{d}t \,\mathrm{d}\xi \,. \end{aligned} \tag{1.20}$$

The inner integral can be adjusted by substituting $z = t(1+|\xi|^2)^{\frac{1}{2}}$, $\mathrm{d}z = (1+|\xi|^2)^{\frac{1}{2}} \,\mathrm{d}t$ to have the form in Lemma 10:

$$\begin{aligned} \int_0^\infty t^{-2s-1} \frac{t^2(1+|\xi|^2)}{1+t^2(1+|\xi|^2)} \,\mathrm{d}t &= \int_0^\infty z^{-2s-1}(1+|\xi|^2)^{\frac{2s+1}{2}} \frac{z^2}{1+z^2} \frac{\mathrm{d}z}{(1+|\xi|^2)^{\frac{1}{2}}} \\ &= (1+|\xi|^2)^s \underbrace{\int_0^\infty \frac{z^{-2s+1}}{1+z^2} \,\mathrm{d}z}_{C} \,. \end{aligned} \tag{1.21}$$

After substituting (1.21) in (1.20) we get:

$$\|u\|_{K,s}^2 = C \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s \,\mathrm{d}\xi = C\|u\|_{H_F^s}^2 \,.$$

□

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