Finite Element Convergence Analysis for non-smooth solutions

Abiy Dejenee Zeleke
Bahru Tsegaye Leyew

Supervisor: Walter Zulehner

Johannes Kepler University Linz, Austria

7th December 2010

Outline

- Finite Element Method
  - Simple 1D Example
- Discretization Error
- Interpolation theorem
- Problem
- K-functional

Outline

- Finite Element Method
  - Simple 1D Example
- Discretization Error
- Interpolation theorem
- Problem
- K-functional

Outline

- Finite Element Method
  - Simple 1D Example
- Discretization Error
- Interpolation theorem
- Problem
- K-functional

Outline

- Finite Element Method
  - Simple 1D Example
- Discretization Error
- Interpolation theorem
- Problem
- K-functional

Outline

- Finite Element Method
  - Simple 1D Example
- Discretization Error
- Interpolation theorem
- Problem
- K-functional

Simple 1D ODE

- Consider the following BVP

$$\begin{cases} -u''(x) = f_\alpha(x), & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $f_\alpha(x) = x^\alpha$, $f_\alpha \in L_2(0,1)$ for $\alpha > -1/2$.

- Exact solution:

$$u(x) = \frac{x - x^{\alpha+2}}{\alpha^2 + 3\alpha + 2}.$$

- Let

$$V = H_0^1(0,1) = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$$

$$u \in V, \quad \int_0^1 u'v'dx = \int_0^1 fv dx, \quad \forall v \in V$$

- Consider the following BVP

$$\begin{cases} -u''(x) = f_\alpha(x), & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $f_\alpha(x) = x^\alpha$, $f_\alpha \in L_2(0,1)$ for $\alpha > -1/2$.

- Exact solution:

$$u(x) = \frac{x - x^{\alpha+2}}{\alpha^2 + 3\alpha + 2}.$$

- Let

$$V = H_0^1(0,1) = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$$

$$u \in V, \quad \int_0^1 u'v'dx = \int_0^1 fv dx, \quad \forall v \in V$$

Simple 1D ODE

- Consider the following BVP

$$\begin{cases} -u''(x) = f_\alpha(x), & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $f_\alpha(x) = x^\alpha$, $f_\alpha \in L_2(0,1)$ for $\alpha > -1/2$.
- Exact solution:

$$u(x) = \frac{x - x^{\alpha+2}}{\alpha^2 + 3\alpha + 2}.$$

- Let

$$V = H_0^1(0,1) = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$$

$$u \in V, \quad \int_0^1 u'v'dx = \int_0^1 fvdx, \quad \forall v \in V$$

- The weak formulation is

$$u \in V, \quad a(u, v) = \ell(v) \quad \forall v \in V \tag{1}$$

- Existence of a unique solution is guaranteed by the Lax-Milgram Lemma
- Partition the domain $\mathrm{I} = [0, 1]$ into N parts as $0 = x_0 < x_1 < ... < x_N = 1$
  - The points $x_i, 0 \leq i \leq N$ are called nodes,
  - The subintervals $\mathrm{I}_i = [x_{i-1}, x_i], 1 \leq i \leq N$ are called elements
    Denote $h_i = x_i - x_{i-1}$ and the mesh parameter $h = max_{1 \leq i \leq N} h_i$
- Approximate solution will be sought in the space $V_h = \{v_h \in V : v_{h|_{\mathrm{I}_i}} \in P_1(\mathrm{I}_i), 1 \leq i \leq N\}$: note that, given the properties of the Sobolev space $H^1(\mathrm{I})$, we also have $v_h \in C(\bar{\mathrm{I}})$

Finite Element Method

- The weak formulation is

$$u \in V, \quad a(u, v) = \ell(v) \quad \forall v \in V \tag{1}$$

- Existence of a unique solution is guaranteed by the Lax-Milgram Lemma
- Partition the domain $\mathrm{I} = [0, 1]$ into N parts as $0 = x_0 < x_1 < ... < x_N = 1$
  - The points $x_i, 0 \leq i \leq N$ are called nodes,
  - The subintervals $\mathrm{I}_i = [x_{i-1}, x_i], 1 \leq i \leq N$ are called elements

    Denote $h_i = x_i - x_{i-1}$ and the mesh parameter $h = max_{1 \leq i \leq N}\, h_i$
- Approximate solution will be sought in the space $V_h = \{v_h \in V : v_{h|_{\mathrm{I}_i}} \in P_1(\mathrm{I}_i), 1 \leq i \leq N\}$: note that, given the properties of the Sobolev space $H^1(\mathrm{I})$, we also have $v_h \in C(\bar{\mathrm{I}})$

Finite Element Method

- The weak formulation is

$$u \in V, \quad a(u,\upsilon) = \ell(\upsilon) \quad \forall \upsilon \in V \tag{1}$$

- Existence of a unique solution is guaranteed by the Lax-Milgram Lemma
- Partition the domain $\mathrm{I} = [0,1]$ into N parts as $0 = x_0 < x_1 < ... < x_N = 1$
  - The points $x_i, 0 \leq i \leq N$ are called nodes,
  - The subintervals $\mathrm{I}_i = [x_{i-1}, x_i], 1 \leq i \leq N$ are called elements
    Denote $h_i = x_i - x_{i-1}$ and the mesh parameter $h = max_{1 \leq i \leq N}\, h_i$
- Approximate solution will be sought in the space $V_h = \{\upsilon_h \in V : \upsilon_{h|_{\mathrm{I}_i}} \in P_1(\mathrm{I}_i), 1 \leq i \leq N\}$: note that, given the properties of the Sobolev space $H^1(\mathrm{I})$, we also have $\upsilon_h \in C(\bar{\mathrm{I}})$

Finite Element Method

- The weak formulation is

$$u \in V, \quad a(u, v) = \ell(v) \quad \forall v \in V \qquad (1)$$

- Existence of a unique solution is guaranteed by the Lax-Milgram Lemma
- Partition the domain $\mathrm{I} = [0, 1]$ into N parts as $0 = x_0 < x_1 < ... < x_N = 1$
  - The points $x_i, 0 \leq i \leq N$ are called nodes,
  - The subintervals $\mathrm{I}_i = [x_{i-1}, x_i], 1 \leq i \leq N$ are called elements
    Denote $h_i = x_i - x_{i-1}$ and the mesh parameter $h = max_{1 \leq i \leq N} h_i$
- Approximate solution will be sought in the space $V_h = \{v_h \in V : v_{h|_{\mathrm{I}_i}} \in P_1(\mathrm{I}_i), 1 \leq i \leq N\}$: note that, given the properties of the Sobolev space $H^1(\mathrm{I})$, we also have $v_h \in C(\bar{\mathrm{I}})$

FEM Continued

- For the basis functions we choose for $i = 1, \ldots, N-1$

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & x_{i-1} \le x \le x_i; \\ \frac{x_{i+1} - x}{h_{i+1}}, & x_i \le x \le x_{i+1}, ; \\ 0, & \text{otherwise.} \end{cases}$$

- The basis functions $V_h = span\{\phi_i, 1 \le i \le N\}$ are linearly independent.

FEM Continued

- For the basis functions we choose for $i = 1, \ldots, N-1$

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & x_{i-1} \le x \le x_i; \\ \frac{x_{i+1} - x}{h_{i+1}}, & x_i \le x \le x_{i+1}, ; \\ 0, & \text{otherwise.} \end{cases}$$

- The basis functions $V_h = span\{\phi_i, 1 \le i \le N\}$ are linearly independent.

- The FEM thus becomes:

$$u_h \in V_h, \quad \int_0^1 (u_h' \upsilon_h') dx = \int_0^1 f_\alpha \upsilon_h dx, \quad \forall \upsilon_h \in V_h$$

- The variational formulation is:

$$u_h \in V_h, \quad a(u_h, \upsilon_h) = \ell(\upsilon_h) \quad \forall \upsilon_h \in V_h \tag{2}$$

which, using the representation $u_h = \sum_{j=1}^{N} z_j \phi_j$, can be transformed to the linear system

$$\sum_{j=1}^{N} z_j \int_0^1 (\phi_i' \phi_j') dx = \int_0^1 f_\alpha \phi_i dx, \quad 1 \le i \le N$$

- This system can be rewritten as $A\mathbf{u} = \mathbf{b}$, where
  - $\mathbf{u} = (z_1, \ldots, z_N)^T$ is the vector of unknown coefficients,
  - $\mathbf{b} = (\int_0^1 f\phi_i dx, \ldots, \int_0^1 f\phi_{N-1} dx, \int_0^1 f\phi_N dx)^T$ is the load vector

FEM Continued

- The FEM thus becomes:

$$u_h \in V_h, \quad \int_0^1 (u_h' \upsilon_h') dx = \int_0^1 f_\alpha \upsilon_h dx, \quad \forall \upsilon_h \in V_h$$

- The variational formulation is:

$$u_h \in V_h, \quad a(u_h, \upsilon_h) = \ell(\upsilon_h) \quad \forall \upsilon_h \in V_h \tag{2}$$

which, using the representation $u_h = \sum_{j=1}^{N} z_j \phi_j$, can be transformed to the linear system

$$\sum_{j=1}^{N} z_j \int_0^1 (\phi_i' \phi_j') dx = \int_0^1 f_\alpha \phi_i dx, \quad 1 \leq i \leq N$$

- This system can be rewritten as $A\mathbf{u} = \mathbf{b}$, where
  - $\mathbf{u} = (z_1, \ldots, z_N)^T$ is the vector of unknown coefficients,
  - $\mathbf{b} = (\int_0^1 f\phi_i dx, \ldots, \int_0^1 f\phi_{N-1} dx, \int_0^1 f\phi_N dx)^T$ is the load vector

FEM Continued

- The FEM thus becomes:

$$u_h \in V_h, \quad \int_0^1 (u_h' \upsilon_h') dx = \int_0^1 f_\alpha \upsilon_h dx, \quad \forall \upsilon_h \in V_h$$

- The variational formulation is:

$$u_h \in V_h, \quad a(u_h, \upsilon_h) = \ell(\upsilon_h) \quad \forall \upsilon_h \in V_h \tag{2}$$

  which, using the representation $u_h = \sum_{j=1}^{N} z_j \phi_j$, can be transformed to the linear system

$$\sum_{j=1}^{N} z_j \int_0^1 (\phi_i' \phi_j') dx = \int_0^1 f_\alpha \phi_i dx, \quad 1 \leq i \leq N$$

- This system can be rewritten as $A\mathbf{u} = \mathbf{b}$, where
  - $\mathbf{u} = (z_1, \ldots, z_N)^T$ is the vector of unknown coefficients,
  - $\mathbf{b} = (\int_0^1 f\phi_i dx, \ldots, \int_0^1 f\phi_{N-1} dx, \int_0^1 f\phi_N dx)^T$ is the load vector

Stiffness Matrix

- Entries of the stiffness matrix can be calculated as $A_{ij} = \int_0^1 (\phi_i' \phi_j') dx$;
  Using the formulae

$$\int_0^1 (\phi_i' \phi_{i-1}') dx = -\frac{1}{h}, \qquad 2 \leq i \leq N,$$

$$\int_0^1 (\phi_i')^2 dx = \frac{2}{h}, \qquad 1 \leq i \leq N,$$

We obtain for the stiffness matrix

$$a(\phi_i, \phi_j) = \begin{cases} \frac{2}{h}, & i = j; \\ -\frac{1}{h}, & |i - j| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

- Entries of the stiffness matrix can be calculated as $A_{ij} = \int_0^1 (\phi_i' \phi_j') dx$;
  Using the formulae

$$\int_0^1 (\phi_i' \phi_{i-1}') dx = -\frac{1}{h}, \qquad 2 \le i \le N,$$

$$\int_0^1 (\phi_i')^2 dx = \frac{2}{h}, \qquad 1 \le i \le N,$$

We obtain for the stiffness matrix

$$a(\phi_i, \phi_j) = \begin{cases} \frac{2}{h}, & i = j; \\ -\frac{1}{h}, & |i-j| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The Discretization Error

Lemma (Céa)

Let the bilinear form $a : H^1 \times H^1 \longrightarrow \mathbb{R}$ be continuous and coercive. Then the solutions $u \in H^1$ and $u_h \in H_h^1$ of the continuous and discrete variational problems satisfy the relation

$$\|u - u_h\|_{H^1} \leq C_0 \|u\|_{H^1}. \tag{3}$$

Idea of proof.

- Use Galerkin Orthogonality $a(u - u_h, v_h) = 0, \forall v_h \in H_h^1$,
- Coercivity of a, i.e. $a(u, u) \geq \gamma \|u\|_{H^1}^2, \forall u \in H_1$, $0 \leq \gamma \leq C \leq \infty$,
- and continuity of a, i.e. $|a(u, v)| \leq C \|u\|_{H^1} \|v\|_{H^1} \forall u, v \in H_1$

□

The Discretization Error (Cont.)

Theorem

Let $u \in H^2(\Omega)$ and u_h be the solutions to the continuous and discrete variational equations respectively. Then for the interpolation error satsfies

$$\|u - u_h\|_{H^1(\Omega)} \leq C_1 h \|u\|_{H^2(\Omega)}. \quad (4)$$

Theorem

Let $u \in H^2(\Omega)$ and u_h be the solutions to the continuous and discrete variational equations respectively. Then for the interpolation error satsfies

$$\|u - u_h\|_{H^1(\Omega)} \leq C_1 h \|u\|_{H^2(\Omega)}. \tag{4}$$

The Discretization Error (Cont.)

Idea of proof.

Since $H^1(0,1) \subset C[0,1]$, we can define an interpolation operator I_h, cont. piecewise linear function, which coincides with v at nodes as $v_h(x_i) = v(x_i)$.
Therefore,

- $\inf_{v_h \in V_h} \|u - u_h\|_{H^1} \leq \|u - I_h u\|_{H^1}$ (Interpolation error).
- Focus on Interpolation error and to get the convergence result use the following steps
  1. Localization of the error on subdomains(Partitions),
  2. Transformation of the subdomains onto the unit interval,
  3. Calculation of the local Interpolation error,
  4. Inverse transformation back to subdomains.

□

Theorem (An Exact Interpolation Theorem)

Let $T : X_0 + X_1 \mapsto Y_0 + Y_1$ *be linear operator with properties*

$$\|Tx\|_{Y_0} \le c_0 \|x\|_{X_0} \qquad \text{and} \qquad \|Tx\|_{Y_1} \le c_1 \|x\|_{X_1} \,,$$

let be either $0 < \theta < 1$ *and* $1 \le q < \infty$ *or* $0 \le \theta \le 1$ *and* $q = \infty$*. Then for* $X_\theta = (X_0,\, X_1)_{\theta,q;K}$ *and* $Y_\theta = (Y_0,\, Y_1)_{\theta,q;K}$

$$\|Tx\|_{Y_\theta} \le c_0^{1-\theta} c_1^{\theta} \|x\|_{X_\theta} \,. \tag{5}$$

The Problem:

Define $T : H^1 \longrightarrow H^1$ by $Tu = u - u_h$ then using equation (3) , (4) and the interpolation theorem, we have:

$$\|u - u_h\|_{H^1} \leq C_0^{1-\theta} C_1^{\theta} h^{\theta} \|u\|_{H^{1+\theta}}. \quad (6)$$

For our model problem the exact solution can be approximated as:- $u(x) \approx x^{\alpha+2}$

$$\int_0^1 |x^{\alpha+2}|^2 dx = \begin{cases} \frac{1}{2\alpha+5}, & \alpha > \frac{-5}{2}; \\ \infty, & else. \end{cases}$$

$$\int_0^1 |x^{\alpha+1}|^2 dx = \begin{cases} \frac{1}{2\alpha+3}, & \alpha > \frac{-3}{2}; \\ \infty, & else. \end{cases}$$

$$\int_0^1 |x^{\alpha}|^2 dx = \begin{cases} \frac{1}{2\alpha+1}, & \alpha > \frac{-1}{2}; \\ \infty, & else. \end{cases}$$

The Problem:

Define $T : H^1 \longrightarrow H^1$ by $Tu = u - u_h$ then using equation (3) , (4) and the interpolation theorem, we have:

$$\|u - u_h\|_{H^1} \leq C_0^{1-\theta} C_1^{\theta} h^{\theta} \|u\|_{H^{1+\theta}}. \tag{6}$$

For our model problem the exact solution can be approximated as:- $u(x) \approx x^{\alpha+2}$

$$\int_0^1 |x^{\alpha+2}|^2 dx = \begin{cases} \frac{1}{2\alpha+5}, & \alpha > \frac{-5}{2}; \\ \infty, & else. \end{cases}$$

$$\int_0^1 |x^{\alpha+1}|^2 dx = \begin{cases} \frac{1}{2\alpha+3}, & \alpha > \frac{-3}{2}; \\ \infty, & else. \end{cases}$$

$$\int_0^1 |x^{\alpha}|^2 dx = \begin{cases} \frac{1}{2\alpha+1}, & \alpha > \frac{-1}{2}; \\ \infty, & else. \end{cases}$$

The K Method

The solution:

$$x^{\alpha+2} \begin{cases} \in H^1, & \alpha > \frac{-3}{2}; \\ \notin H^2, & \alpha \in (\frac{-3}{2}, \frac{-1}{2}]. \end{cases}$$

Equivalent to: $v := u'(x) \approx x^\gamma$ for $\gamma = \alpha + 1$

$$x^{\gamma} \begin{cases} \in L_2, & \alpha > \frac{-3}{2}; \\ \notin H^1, & \alpha \in (\frac{-3}{2}, \frac{-1}{2}]. \end{cases}$$

Now we need to find θ so that $x^\gamma \in H^\theta$ for $\gamma \in (-\frac{1}{2}, \frac{1}{2})$. $x^\gamma \in H^\theta$ if

$$\int_0^\infty t^{-(2\theta+1)} K(t; v)^2 dt < \infty$$

with

$$K(t; v)^2 = \inf_{v = v_0 + v_1} (\|v_0\|_{L_2}^2 + t^2 \|v_1\|_{H_1}^2).$$

We know $K(t; v) \leq \|v\|_{L_2}$.

The solution:

$$x^{\alpha+2}\begin{cases}\in H^1, & \alpha > \frac{-3}{2};\\ \notin H^2, & \alpha \in (\frac{-3}{2}, \frac{-1}{2}].\end{cases}$$

Equivalent to: $v := u'(x) \approx x^\gamma$ for $\gamma = \alpha + 1$

$$x^{\gamma}\begin{cases}\in L_2, & \alpha > \frac{-3}{2};\\ \notin H^1, & \alpha \in (\frac{-3}{2}, \frac{-1}{2}].\end{cases}$$

Now we need to find θ so that $x^\gamma \in H^\theta$ for $\gamma \in (-\frac{1}{2}, \frac{1}{2})$. $x^\gamma \in H^\theta$ if

$$\int_0^\infty t^{-(2\theta+1)} K(t; v)^2 dt < \infty$$

with

$$K(t; v)^2 = \inf_{v=v_0+v_1} (\|v_0\|_{L_2}^2 + t^2 \|v_1\|_{H_1}^2).$$

We know $K(t; v) \leq \|v\|_{L_2}$.

The K Method

The solution:

$$x^{\alpha+2}\begin{cases}\in H^1, & \alpha > \frac{-3}{2};\\ \notin H^2, & \alpha\in(\frac{-3}{2},\frac{-1}{2}].\end{cases}$$

Equivalent to: $v := u'(x) \approx x^\gamma$ for $\gamma = \alpha + 1$

$$x^{\gamma}\begin{cases}\in L_2, & \alpha > \frac{-3}{2};\\ \notin H^1, & \alpha\in(\frac{-3}{2},\frac{-1}{2}].\end{cases}$$

Now we need to find θ so that $x^\gamma \in H^\theta$ for $\gamma \in (-\frac{1}{2}, \frac{1}{2})$. $x^\gamma \in H^\theta$ if

$$\int_0^\infty t^{-(2\theta+1)} K(t;v)^2 dt < \infty$$

with

$$K(t;v)^2 = \inf_{v=v_0+v_1} (\|v_0\|_{L_2}^2 + t^2\|v_1\|_{H_1}^2).$$

We know $K(t;v) \leq \|v\|_{L_2}$.

The K Method

The solution:

$$x^{\alpha+2} \begin{cases} \in H^1, & \alpha > \frac{-3}{2}; \\ \notin H^2, & \alpha \in (\frac{-3}{2}, \frac{-1}{2}]. \end{cases}$$

Equivalent to: $v := u^{'}(x) \approx x^{\gamma}$ for $\gamma = \alpha + 1$

$$x^{\gamma} \begin{cases} \in L_2, & \alpha > \frac{-3}{2}; \\ \notin H^1, & \alpha \in (\frac{-3}{2}, \frac{-1}{2}]. \end{cases}$$

Now we need to find θ so that $x^{\gamma} \in H^{\theta}$ for $\gamma \in (-\frac{1}{2}, \frac{1}{2})$. $x^{\gamma} \in H^{\theta}$ if

$$\int_0^{\infty} t^{-(2\theta+1)} K(t; v)^2 dt < \infty$$

with

$$K(t; v)^2 = \inf_{v=v_0+v_1} (\|v_0\|_{L_2}^2 + t^2 \|v_1\|_{H_1}^2).$$

We know $K(t; v) \leq \|v\|_{L_2}$.

K-Method(cont.)

we focus on

$$\int_0^1 t^{-(2\theta+1)} K(t;v)^2 dt < \infty$$

Decompose $v(x) = v_0(x) + v_1(x)$ for $\varepsilon \in (0, \frac{1}{2})$

$$v_0(x) = max(x^\gamma - \varepsilon^\gamma, 0)$$
$$v_1(x) = min(x^\gamma, \varepsilon^\gamma)$$

$$\|v_0\|_{L_2}^2 + t^2 \|v_1\|_{H_1}^2 = \int_0^1 |v_0(x)|^2 dx + t^2 \int_0^1 (|v_1(x)|^2 + |v_1'(x)|^2) dx$$

$$g(t;\varepsilon) := \frac{2\gamma}{\gamma+1}\varepsilon^{2\gamma+1} - \frac{\gamma^2}{2\gamma-1} t^2 \varepsilon^{2\gamma-1} + C$$

K-Method(cont.)

we focus on

$$\int_0^1 t^{-(2\theta+1)} K(t;v)^2 dt < \infty$$

Decompose $v(x) = v_0(x) + v_1(x)$ for $\varepsilon \in (0, \frac{1}{2})$

$$v_0(x) = max(x^\gamma - \varepsilon^\gamma, 0)$$
$$v_1(x) = min(x^\gamma, \varepsilon^\gamma)$$

$$\|v_0\|_{L_2}^2 + t^2 \|v_1\|_{H_1}^2 = \int_0^1 |v_0(x)|^2 dx + t^2 \int_0^1 (|v_1(x)|^2 + |v_1'(x)|^2) dx$$
$$g(t;\varepsilon) := \frac{2\gamma}{\gamma+1}\varepsilon^{2\gamma+1} - \frac{\gamma^2}{2\gamma-1} t^2 \varepsilon^{2\gamma-1} + C$$

Minimize $g(t;\varepsilon)$ over ε.

$$g^{'}(t;\varepsilon) = \frac{2(2\gamma+1)\gamma}{\gamma+1}\varepsilon^{2\gamma} - \gamma^2 t^2 \varepsilon^{2\gamma-2} = 0$$

gives

$$\varepsilon = t\sqrt{\frac{(\gamma+1)\gamma}{2(2\gamma+1)}} = C_1 t$$

$$g^{''}(t;\varepsilon) = \frac{4(2\gamma+1)\gamma^2}{\gamma+1}\varepsilon^{2\gamma-1} - 2\gamma^2(\gamma-1)t^2\varepsilon^{2\gamma-3} \geq 0.$$

Therefore

$$\inf_{\varepsilon} g(t;\varepsilon) = \frac{2\gamma}{\gamma+1)}(C_1 t)^{2\gamma+1} - \frac{\gamma^2}{2\gamma-1}t^2(C_1 t)^{2\gamma-1} + C = C_2 t^{2\gamma+1}$$

Hence

$$K(t;v)^2 \leq C_2 t^{2\alpha+3}.$$

K-Method(cont.)

Minimize $g(t;\varepsilon)$ over ε.

$$g^{'}(t;\varepsilon) = \frac{2(2\gamma+1)\gamma}{\gamma+1}\varepsilon^{2\gamma} - \gamma^2 t^2 \varepsilon^{2\gamma-2} = 0$$

gives

$$\varepsilon = t\sqrt{\frac{(\gamma+1)\gamma}{2(2\gamma+1)}} = C_1 t$$

$$g^{''}(t;\varepsilon) = \frac{4(2\gamma+1)\gamma^2}{\gamma+1}\varepsilon^{2\gamma-1} - 2\gamma^2(\gamma-1)t^2\varepsilon^{2\gamma-3} \geq 0.$$

Therefore

$$inf_{\varepsilon}\, g(t;\varepsilon) = \frac{2\gamma}{\gamma+1)}(C_1 t)^{2\gamma+1} - \frac{\gamma^2}{2\gamma-1}t^2(C_1 t)^{2\gamma-1} + C = C_2 t^{2\gamma+1}$$

Hence

$$K(t;v)^2 \leq C_2 t^{2\alpha+3}.$$

Minimize $g(t;\varepsilon)$ over ε.

$$g'(t;\varepsilon) = \frac{2(2\gamma+1)\gamma}{\gamma+1}\varepsilon^{2\gamma} - \gamma^2 t^2 \varepsilon^{2\gamma-2} = 0$$

gives

$$\varepsilon = t\sqrt{\frac{(\gamma+1)\gamma}{2(2\gamma+1)}} = C_1 t$$

$$g''(t;\varepsilon) = \frac{4(2\gamma+1)\gamma^2}{\gamma+1}\varepsilon^{2\gamma-1} - 2\gamma^2(\gamma-1)t^2\varepsilon^{2\gamma-3} \geq 0.$$

Therefore

$$\inf_{\varepsilon} g(t;\varepsilon) = \frac{2\gamma}{\gamma+1)}(C_1 t)^{2\gamma+1} - \frac{\gamma^2}{2\gamma-1}t^2(C_1 t)^{2\gamma-1} + C = C_2 t^{2\gamma+1}$$

Hence

$$K(t;v)^2 \leq C_2 t^{2\alpha+3}.$$

Minimize $g(t;\varepsilon)$ over ε.

$$g'(t;\varepsilon) = \frac{2(2\gamma+1)\gamma}{\gamma+1}\varepsilon^{2\gamma} - \gamma^2 t^2 \varepsilon^{2\gamma-2} = 0$$

gives

$$\varepsilon = t\sqrt{\frac{(\gamma+1)\gamma}{2(2\gamma+1)}} = C_1 t$$

$$g''(t;\varepsilon) = \frac{4(2\gamma+1)\gamma^2}{\gamma+1}\varepsilon^{2\gamma-1} - 2\gamma^2(\gamma-1)t^2\varepsilon^{2\gamma-3} \geq 0.$$

Therefore

$$\inf_{\varepsilon} g(t;\varepsilon) = \frac{2\gamma}{\gamma+1)}(C_1 t)^{2\gamma+1} - \frac{\gamma^2}{2\gamma-1}t^2(C_1 t)^{2\gamma-1} + C = C_2 t^{2\gamma+1}$$

Hence

$$K(t;v)^2 \leq C_2 t^{2\alpha+3}.$$

Minimize $g(t;\varepsilon)$ over ε.

$$g^{'}(t;\varepsilon) = \frac{2(2\gamma+1)\gamma}{\gamma+1}\varepsilon^{2\gamma} - \gamma^2 t^2 \varepsilon^{2\gamma-2} = 0$$

gives

$$\varepsilon = t\sqrt{\frac{(\gamma+1)\gamma}{2(2\gamma+1)}} = C_1 t$$

$$g^{''}(t;\varepsilon) = \frac{4(2\gamma+1)\gamma^2}{\gamma+1}\varepsilon^{2\gamma-1} - 2\gamma^2(\gamma-1)t^2\varepsilon^{2\gamma-3} \geq 0.$$

Therefore

$$\inf_{\varepsilon} g(t;\varepsilon) = \frac{2\gamma}{\gamma+1)}(C_1 t)^{2\gamma+1} - \frac{\gamma^2}{2\gamma-1}t^2(C_1 t)^{2\gamma-1} + C = C_2 t^{2\gamma+1}$$

Hence

$$K(t;v)^2 \leq C_2 t^{2\alpha+3}.$$

Application of Interpolation theorem

Then

$$\int_0^1 t^{-(2\theta+1)} K(t;v)^2 dt \leq \int_0^1 t^{2(\alpha-\theta+1)} dt < \infty,$$

if $2(\alpha - \theta + 1) > -1 \implies \theta < \frac{3}{2} + \alpha$.

This implies

$$v(x) = x^{\alpha+1} \in H^{\theta}, \text{ for } \theta < \frac{3}{2} + \alpha,$$

$$\implies u(x) = x^{\alpha+2} \in H^{\frac{5}{2}+\alpha}, \text{ for } \alpha \in (-\frac{3}{2}, -\frac{1}{2}).$$

Then we can estimate the error as $h \to 0$ in

$$\|u - u_h\|_{H^1} \leq C_0^{1-\theta} C_1^{\theta} h^{\theta} \|u\|_{H^{1+\theta}}. \tag{7}$$

Application of Interpolation theorem

Then

$$\int_0^1 t^{-(2\theta+1)} K(t;v)^2 dt \leq \int_0^1 t^{2(\alpha-\theta+1)} dt < \infty,$$

if $2(\alpha - \theta + 1) > -1 \implies \theta < \frac{3}{2} + \alpha$.
This implies

$$v(x) = x^{\alpha+1} \in H^{\theta}, \text{ for } \theta < \frac{3}{2} + \alpha,$$
$$\implies u(x) = x^{\alpha+2} \in H^{\frac{5}{2}+\alpha}, \text{ for } \alpha \in (-\frac{3}{2}, -\frac{1}{2}).$$

Then we can estimate the error as $h \to 0$ in

$$\|u - u_h\|_{H^1} \leq C_0^{1-\theta} C_1^{\theta} h^{\theta} \|u\|_{H^{1+\theta}}. \tag{7}$$

Then

$$\int_0^1 t^{-(2\theta+1)} K(t;v)^2 dt \le \int_0^1 t^{2(\alpha-\theta+1)} dt < \infty,$$

if $2(\alpha-\theta+1) > -1 \implies \theta < \frac{3}{2}+\alpha$.
This implies

$$v(x) = x^{\alpha+1} \in H^{\theta}, \text{ for } \theta < \frac{3}{2}+\alpha,$$
$$\implies u(x) = x^{\alpha+2} \in H^{\frac{5}{2}+\alpha}, \text{ for } \alpha \in (-\frac{3}{2}, -\frac{1}{2}).$$

Then we can estimate the error as $h \to 0$ in

$$\|u-u_h\|_{H^1} \le C_0^{1-\theta} C_1^{\theta} h^{\theta} \|u\|_{H^{1+\theta}}. \tag{7}$$

Thank you!