Interpolation in $\mathbb{R}^n$

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19 October 2010

Outline

- Matrix Functions
- Inner Products
- The Dual Norm
- Sum and Intersection
- First Interpolation Theorem
- Generalized Eigenvalue Problem
- The Norm $\|.\|_\theta$
  1. First Representation
  2. Second Representation
- Interpolation Theorem (cont.)

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Let M be a diagonalizable matrix

$$M = XDX^{-1} \quad \text{with} \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

X non-singular matrix.
Let f be a scalar function well defined on the spectrum of M.
Set

$$f(M) = Xf(D)X^{-1} \quad \text{with} \quad f(D) = \begin{bmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{bmatrix},$$

$f(M)$ is well defined.

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Inner Products

Let $X = (\mathbb{R}^n, (.,.)_X)$ be a Hilbert space with inner product

$$(x, y)_X = \sum_{i,j=1}^{n} (e_i, e_j)_X x_i y_j.$$

M is given by

$$M = (M_{ij}) \text{ with } M_{ij} = (e_i, e_j)_X,$$

and we have

$$(x, y)_X = y^T M x = \langle Mx, y \rangle.$$

In general, M, symmetric and positive definite, defines

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Let $f \in \mathbb{R}^n$.
For the dual norm

$$\|f\|_{X^*} := \sup_{x \in X} \frac{\langle f, x \rangle}{\|x\|_X}$$

$$\Rightarrow \|f\|^2_{X^*} = \sup_{x \in X} \frac{\langle f, x \rangle^2}{\langle Mx, x \rangle} = \sup_{x \in X} \frac{\langle f, M^{-1/2}x \rangle^2}{\langle M^{-1/2}Mx, M^{-1/2}x \rangle}$$

$$= \sup_{x \in X} \frac{\langle f, M^{-1/2}x \rangle^2}{\langle x, x \rangle} = \langle M^{-1/2}f, M^{-1/2}f \rangle$$

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Therefore, $X^* = (\mathbb{R}^n, (.,.)_{X^*})$ is dual Hilbert space.
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Let M_0 and M_1 be symmetric and positive definite matrices defining

$$(x, y)_{X_0} = \langle M_0 x, y\rangle, \quad (x, y)_{X_1} = \langle M_1 x, y\rangle$$

Hilbert spaces $X_0 = (\mathbb{R}^n, (.,.)_{X_0})$ and $X_1 = (\mathbb{R}^n, (.,.)_{X_1})$.
Norms of $X_0 + X_1$ and $X_0 \cap X_1$ given by

$$\|x\|^2_{X_0+X_1} = \inf_{x=x_0+x_1}(\|x_0\|^2_{X_0} + \|x_1\|^2_{X_1}),$$

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Sum and Intersection

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Associated with

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\begin{aligned}
(x,y)_{X_0+X_1} &= \langle (M_0^{-1}+M_1^{-1})^{-1}x, y\rangle \\
(x,y)_{X_0\cap X_1} &= \langle (M_0+M_1)x, y\rangle
\end{aligned} \tag{1}
$$

Hilbert spaces $X_0+X_1=(\mathbb{R}^n,(.,.)_{X_0+X_1})$ and $X_0\cap X_1=(\mathbb{R}^n,(.,.)_{X_0\cap X_1})$.
From (1) follows

$$
\begin{aligned}
(x,y)_{(X_0+X_1)^*} &= (x,y)_{X_0^*\cap X_1^*} \\
(x,y)_{(X_0\cap X_1)^*} &= (x,y)_{X_0^*+X_1^*}
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Hence,

$$(X_0+X_1)^* = X_0^*\cap X_1^* \text{ and } (X_0\cap X_1)^* = X_0^*+X_1^*.$$

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Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ *with*

$$\|Tx\|_{Y_0} \leq c_0\|x\|_{X_0} \text{ and } \|Tx\|_{Y_1} \leq c_1\|x\|_{X_1},$$

where the norms $\|.\|_{X_i}$ *and* $\|.\|_{Y_i}$ *are the norms associated to the inner products:*

$$(x, y)_{X_i} = \langle M_i x, y\rangle \text{ and } (x, y)_{Y_i} = \langle N_i x, y\rangle \text{ for i=0,1.}$$

Then,

$$\|Tx\|_{Y_0+Y_1} \leq max(c_0, c_1)\|x\|_{X_0+X_1},$$

and

$$\|Tx\|_{Y_0\cap Y_1} \leq max(c_0, c_1)\|x\|_{X_0\cap X_1}.$$

Interpolation Theorem

Proof.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ with

$$
\begin{aligned}
\|Tx\|^2_{Y_0+Y_1} &= \inf_{Tx=y_0+y_1} (\|y_0\|^2_{Y_0} + \|y_1\|^2_{Y_1}), \\
&\leq \inf_{x=x_0+x_1} (\|Tx_0\|^2_{Y_0} + \|Tx_1\|^2_{Y_1}), (Particular \quad decomposition) \\
&\leq \inf_{x=x_0+x_1} (c_0^2\|x_0\|^2_{X_0} + c_1^2\|x_1\|^2_{X_1}), \\
&\leq max(c_0^2, c_1^2) \inf_{x=x_0+x_1} (\|x_0\|^2_{X_0} + \|x_1\|^2_{X_1}), \\
&= max(c_0^2, c_1^2)\|x\|^2_{X_0+X_1}.
\end{aligned}
$$

□

The proof of the second estimate is straight forward.

Generalized eigenvalue problem:

Consider

$$M_0^{-1}M_1 x = \lambda x$$

$M_0^{-1}M_1$ self adjoint and positive definite w.r.t $(.,.)_{X_0}$.
$\{e_i : i = 1, 2, ..., n\}$ is an orthonormal basis of eigenvectors with $(e_i, e_j)_{X_0} = \delta_{ij}$.
Each vector $x \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n \hat{x}_i e_i$.
Then

$$\|x\|_{X_0}^2 = \sum_{i=1}^n \hat{x}_i^2 \text{ and } \|x\|_{X_1}^2 = \sum_{i=1}^n \lambda_i \hat{x}_i^2.$$

Moreover,

$$\|x\|_{X_0+X_1}^2 = \sum_{i=1}^n \frac{\lambda_i}{1+\lambda_i}\hat{x}_i^2 \text{ and } \|x\|_{X_0\cap X_1}^2 = \sum_{i=1}^n (1+\lambda_i)\hat{x}_i^2.$$

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Justification

We justify the sum and intersection norms as

$$\begin{aligned}
\|x\|^2_{X_0+X_1} &= (x,x)_{X_0+X_1} \\
&= \langle (M_0^{-1}+M_1^{-1})^{-1}x, x\rangle \\
&= \langle (M_0^{-1}+\lambda^{-1}M_0^{-1})^{-1}x, x\rangle \\
&= \langle (1+\lambda^{-1})M_0 x, x\rangle \\
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The justification for intersection norm is forward.

The norm $\|.\|_\theta$

For each $\theta \in [0, 1]$, introduce a norm by

$$\|x\|_\theta^2 = \sum_{i=1}^{n} \lambda_i^\theta \hat{x}_i^2. \qquad (2)$$

Observe

$$\|x\|_0 = \|x\|_{X_0} \text{ and } \|x\|_1 = \|x\|_{X_1}.$$

So, we have introduced normed space $X_\theta = (\mathbb{R}^n, \|.\|_\theta)$.

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First representation of the norm $\|.\|_\theta$

We have

$$(M_0^{-1/2} M_1 M_0^{-1/2})(M_0^{1/2} e_i) = \lambda_i (M_0^{1/2} e_i).$$

Then

$$\langle (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2} e_i, M_0^{1/2} e_j \rangle$$
$$= \lambda_i^\theta \langle M_0^{1/2} e_i, M_0^{1/2} e_j \rangle = \lambda_i^\theta \delta_{ij}.$$

Implies

$$\sum_{i=1}^{n} \lambda_i^\theta \hat{x}_i^2 = \langle (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2} x, M_0^{1/2} x \rangle$$

$\|.\|_{X_\theta}$ associated norm to the inner product

$$(x, y)_{X_\theta} = \langle M_\theta x, y \rangle \text{ with } M_\theta = M_0^{1/2} (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2},$$

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$$(x, y)_{X_\theta} = \langle M_\theta x, y \rangle \text{ with } M_\theta = M_0^{1/2} (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2},$$

Hilbert space $X_\theta = (\mathbb{R}^n, (.,.))_{X_\theta}$

We have

$$(M_0^{-1/2} M_1 M_0^{-1/2})(M_0^{1/2} e_i) = \lambda_i (M_0^{1/2} e_i).$$

Then

$$\begin{aligned}\langle (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2} e_i, M_0^{1/2} e_j \rangle \\ = \lambda_i^\theta \langle M_0^{1/2} e_i, M_0^{1/2} e_j \rangle = \lambda_i^\theta \delta_{ij}.\end{aligned}$$

Implies

$$\sum_{i=1}^{n} \lambda_i^\theta \hat{x}_i^2 = \langle (M_0^{-1/2} M_1 M_0^{-1/2})^\theta M_0^{1/2} x, M_0^{1/2} x \rangle$$

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Second Representation of the norm

Let $\theta \in (0, 1)$.

Then, from equation (2), definition of $\|.\|_{X_\theta}$, and identity

$$\int_0^\infty \frac{t^{-(2\theta+1)}}{1+t^{-2}\lambda_i^{-1}}dt = \lambda_i^\theta \int_0^\infty \frac{s^{1-2\theta}}{1+s^2}ds = \lambda_i^\theta c_\theta,$$

(Substitution rule for $s = \sqrt{\lambda_i}t$)

where

$$c_\theta = \frac{\pi}{2sin(\theta\pi)},$$

Then

$$\|x\|_{X_\theta}^2 = \sum_{i=1}^n \lambda_i^\theta \hat{x}_i^2 = c_\theta^{-1} \sum_{i=1}^n \int_0^\infty \frac{t^{-(2\theta+1)}}{1+t^{-2}\lambda_i^{-1}} dt \hat{x}_i^2,$$

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(Substitution rule for $s = \sqrt{\lambda_i}t$)
where

$$c_\theta = \frac{\pi}{2sin(\theta\pi)},$$

Then

$$\|x\|^2_{X_\theta} = \sum_{i=1}^n \lambda_i^\theta \hat{x}_i{}^2 = c_\theta^{-1} \sum_{i=1}^n \int_0^\infty \frac{t^{-(2\theta+1)}}{1+t^{-2}\lambda_i^{-1}} dt \hat{x}_i{}^2,$$

$$= c_\theta^{-1} \int_0^\infty t^{-(2\theta+1)} \sum_{i=1}^n \frac{1}{1+t^{-2}\lambda_i^{-1}} \hat{x}_i{}^2 dt.$$

$$\sum_{i=1}^{n} \frac{1}{1+t^{-2}\lambda_i^{-1}} \hat{x}_i{}^2 = \sum_{i=1}^{n} \frac{t^2\lambda_i}{1+t^2\lambda_i} \hat{x}_i{}^2 = \langle (M_0^{-1} + t^{-2}M_1^{-1})^{-1} x, x \rangle$$
$$= \inf_{x=x_0+x_1} (\|x_0\|_{X_0}^2 + t^2 \|x_1\|_{X_1}^2)$$

Therefore,

$$\|x\|_\theta^2 = c_\theta^{-1} \int_0^\infty t^{-(2\theta+1)} K(t;x)^2 dt$$

with

$$K(t;x) = \inf_{x=x_0+x_1} (\|x_1\|_{X_0}^2 + t^2 \|x_1\|_{X_1}^2)^{1/2}.$$

Second Representation of the norm (cont.)

$$\sum_{i=1}^{n} \frac{1}{1+t^{-2}\lambda_i^{-1}}\hat{x}_i{}^2 = \sum_{i=1}^{n} \frac{t^2\lambda_i}{1+t^2\lambda_i}\hat{x}_i{}^2 = \langle (M_0^{-1}+t^{-2}M_1^{-1})^{-1}x, x\rangle$$
$$= \inf_{x=x_0+x_1}(\|x_0\|_{X_0}^2 + t^2\|x_1\|_{X_1}^2)$$

Therefore,

$$\|x\|_\theta^2 = c_\theta^{-1}\int_0^\infty t^{-(2\theta+1)}K(t;x)^2dt$$

with

$$K(t;x) = \inf_{x=x_0+x_1}(\|x_1\|_{X_0}^2 + t^2\|x_1\|_{X_1}^2)^{1/2}.$$

$$\sum_{i=1}^{n} \frac{1}{1+t^{-2}\lambda_i^{-1}}\hat{x}_i{}^2 = \sum_{i=1}^{n} \frac{t^2\lambda_i}{1+t^2\lambda_i}\hat{x}_i{}^2 = \langle (M_0^{-1} + t^{-2}M_1^{-1})^{-1}x, x\rangle$$
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Therefore,

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Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ *with*

$$\|Tx\|_{Y_0} \leq c_0 \|x\|_{X_0} \text{ and } \|Tx\|_{Y_1} \leq c_1 \|x\|_{X_1},$$

where the norms $\|.\|_{X_i}$ *and* $\|.\|_{Y_i}$ *are the norms associated to the inner products:*

$$(x, y)_{X_i} = \langle M_i x, y \rangle \text{ and } (x, y)_{Y_i} = \langle N_i x, y \rangle \text{ for i=0,1.}$$

Then, for $X_\theta = [X_0, X_1]_\theta$ *and* $Y_\theta = [Y_0, Y_1]_\theta$*, we have*

$$\|Tx\|_{Y_\theta} \leq c_0^{1-\theta} c_1^{\theta} \|x\|_{X_\theta}.$$

Interpolation Theorem (cont.)

Proof.

$$\begin{aligned}
\|Tx\|_{Y_\theta}^2 &= c_\theta^{-1}\int_0^\infty t^{-2\theta-1}\inf_{Tx=y_0+y_1}(\|y_0\|_{Y_0}^2+t^2\|y_1\|_{Y_1}^2)dt,\\
&\le c_\theta^{-1}\int_0^\infty t^{-2\theta-1}\inf_{x=x_0+x_1}(\|Tx_0\|_{Y_0}^2+t^2\|Tx_1\|_{Y_1}^2)dt,\\
&(\textit{Particular}\quad \textit{decomposition.})\\
&\le c_\theta^{-1}\int_0^\infty t^{-2\theta-1}\inf_{x=x_0+x_1}(c_0^2\|x_0\|_{X_0}^2+c_1^2t^2\|x_1\|_{X_1}^2)dt,\\
&= c_0^2c_\theta^{-1}\int_0^\infty t^{-2\theta-1)}\inf_{x=x_0+x_1}\left(\|x_0\|_{X_0}^2+\left(\frac{c_1}{c_0}\right)^2t^2\|x_1\|_{X_1}^2\right)dt,\\
&= c_0^2\left(\frac{c_1}{c_0}\right)^{2\theta}c_\theta^{-1}\int_0^\infty s^{-2\theta-1}\inf_{x=x_0+x_1}(\|x_0\|_{X_0}^2+s^2\|x_1\|_{X_1}^2)ds,\\
&= c_0^{2(1-\theta)}c_1^{2\theta}\|x\|_{X_\theta}^2.
\end{aligned}$$

Thank you!