Multilevel Representations, Space Interpolation and Preconditioning

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Existence and uniqueness of a variational problem

We consider the **variational problem**:

Find $u \in H_0^1(\Omega)$ such that

$$a(u, \varphi) = \langle f, \varphi \rangle \quad \forall \varphi \in H_0^1(\Omega) =: V_0 \qquad (1)$$

for a given $f \in (H_0^1(\Omega))^* = H^{-1}(\Omega)$.

In the following, we assume that the bilinear form

$$a(\cdot, \cdot) : V_0 \times V_0 \to \mathbb{R}$$

is **symmetric**, V_0-**elliptic** and V_0-**bounded**, which guarantees the existence of a unique solution of the variational problem (Lax-Milgram theorem).

Galerkin approximation

Let $M_h \subset H_0^1(\Omega)$ be a finite dimensional subspace. Then we have the Galerkin scheme:

Find $u_h \in M_h$ such that

$$a(u_h, \varphi_h) = \langle f, \varphi_h \rangle \quad \forall \varphi_h \in M_h =: M.$$

Let $\{\varphi_i\}_{i=\overline{1,\dim M_h}}$ be a basis for M_h. Inserting the representation

$$u_h = \sum u_i \varphi_i$$

into the equation above leads to

$$\begin{aligned} \sum u_i a(\varphi_i, \varphi_j) &= \langle f, \varphi_j \rangle, \quad j = \overline{1, \dim M_h} \\ \underline{\underline{A}}_h \underline{u}_h &= \underline{f}_h. \end{aligned}$$

Spectral condition number and preconditioning

Due to our assumptions on the bilinear form $a(\cdot,\cdot)$, our **matrix** $\underline{\underline{A}}_h$ is symmetric positive definite and we get the following (**basis dependent**) condition number by choosing a nodal FE-basis:

$$\kappa(\underline{\underline{A}}_h) = cond_2(\underline{\underline{A}}_h) = \frac{\lambda_{\max}(\underline{\underline{A}}_h)}{\lambda_{\min}(\underline{\underline{A}}_h)} = O(h^{-2}).$$

Other approach: Define the symmetric positive definite (SPD) **operator** $A_h : M_h \longrightarrow M_h^*$ by

$$(A_h u_h, \varphi_h)_{L_2(\Omega)} = a(u_h, \varphi_h) \quad \forall u_h, \varphi_h \in M_h$$

and the (**basis independent**) operator condition number is

$$\kappa(A_h) = \frac{\lambda_{\max}(A_h)}{\lambda_{\min}(A_h)} = O(h^{-2}).$$

What is a preconditioner?

Let $B_h^{-1} : M_h^* \to M_h$ be another SPD operator.
A "good preconditioner" should have the following properties:

1. the action of B_h^{-1} on M_h^* is "cheap"
2. the condition number $\kappa(B_h^{-1}A_h) << \kappa(A_h)$

Two extreme cases:

- $B_h^{-1} = I_h$ - cheap but property 2 is not valid
- $B_h^{-1} = A_h^{-1}$ - we have $\kappa(B_h^{-1}A_h) = \kappa(I_h) = 1$ but not cheap

New notation: $A_h = A, B_h = B$

Proposition

Suppose there exist constants $c_1, c_2 > 0$ with

$$c_1(Bv, v) \le (Av, v) \le c_2(Bv, v) \quad \forall v \in M. \tag{2}$$

Then $\kappa(B^{-1}A) \le c_2/c_1$.

Multilevel setting

Splitting Ω in a multilevel fashion:

derive τ_{K+1} by deviding each triangle ($\triangle$) in τ_K into 4 congruent triangles (called **dyadic decomposition**)

for each τ_K we define $h_K := \max_{\triangle \in \tau_K} \operatorname{diam} \triangle$

$$\Rightarrow M_1 \subset M_2 \subset \ldots \subset M_K \subset \cdots \subset H^1(\Omega) =: V$$

properties of M_K:

- M_K consists of $\mathcal{C}^0$, piecewise linear functions on τ_K
- M_K is a finite dimensional, linear space, and any formulation defined on V is also well-defined on M_K

L_2-projections

Let us define L_2-projections $Q_K : L_2(\Omega) \to M_K$ by the identity

$$(Q_K u, \varphi) = (u, \varphi) \quad \forall \varphi \in M_K \;\; \forall u \in L_2(\Omega), \tag{3}$$

and additionally, it should be valid

$$(Q_K u, \varphi) = \langle u, \varphi \rangle \quad \forall \varphi \in M_K \;\; \forall u \in V^*,$$

where we have the notation:
$\| \cdot \| = \| \cdot \|_0 = \| \cdot \|_{L_2(\Omega)}$, $(\cdot, \cdot) = (\cdot, \cdot)_0 = (\cdot, \cdot)_{L_2(\Omega)}$.

Jackson's (approximation) inequalities

Lemma (Jackson's inequalities)

1. $\|(I - Q_K)u\| \leq c_2 h_K^2 \|u\|_{H^2(\Omega)} \qquad \forall u \in H^2(\Omega)$
2. $\|(I - Q_K)u\| \leq 1 \cdot \|u\| \qquad \forall u \in L_2(\Omega)$
3. $\|(I - Q_K)u\| \leq c_2^s h_K^{2s} \|u\|_{H^{2s}(\Omega)} \qquad \forall u \in H^{2s}(\Omega)$

   with $H^{2s}(\Omega) = [L_2(\Omega), H^2(\Omega)]_s$, $0 < s < 1$.

Proof

Definitions and Remarks

Let $0 < h \leq h_0$ be a fixed parameter and τ_h be a family of triangulations of Ω:

$$U(h) := \frac{h}{\underline{h}} = \frac{\max_{\tau \in \tau_h} h_\tau}{\min_{\tau \in \tau_h} h_s} = \text{uniformity number},$$

where h_τ is the edge length of a triangle and h_s is the sphere diameter of the incircle of a triangle.

Definition

The family τ_h is called quasi-uniform if

$$U(h) \leq const \quad \forall h : 0 < h \leq h_0.$$

Remark: We have a mapping to a reference element (unit triangle).

Statement

Again: $M_1 \subset M_2 \subset ... \subset M_K \subset \cdots \subset H^1(\Omega) =: V$
is a quasi-uniform family of nested dyadic refinements.

We will show:

$$\boxed{M_K \subset H^s(\Omega) \qquad \text{for } 0 \leq s \leq \gamma < \frac{3}{2},} \tag{4}$$

where $H^0(\Omega) = L_2(\Omega)$, $H^1(\Omega)$, $H^s(\Omega) = [L_2(\Omega), H^1(\Omega)]_s$ for $s \in (0,1)$ and $H^s(\Omega) = [L_2(\Omega), H^2(\Omega)]_{s/2}$ for $s \in (0,2)$ with

$$\|v\|_s^2 = \|v\|_{H^s(\Omega)}^2 := \|v\|_1^2 + \sum_{|\alpha|=1} \|D^\alpha v\|_\beta^2$$

with $s = 1 + \beta$, $0 < \beta < \frac{1}{2}$, but $M_K \not\subset H^{\frac{3}{2}}(\Omega)$, i.e.
$M_K \subset H^{\frac{3}{2}-\epsilon}(\Omega)$ for any small $\epsilon > 0$.

Convexity inequality

Theorem (Logarithmic convexity inequality)

$$\|v\|_s = \|v\|_{H^s(\Omega)=[L_2,H^1]_s} \leq c_{s,2}\|v\|_0^{1-s}\|v\|_1^s \quad \forall v \in H^1(\Omega)$$

with $c_{s,2} = \frac{1}{\sqrt{2s(1-s)}}$.

The inequality follows from a more general result.

Convexity inequality

Theorem (Convexity inequality in general)

For $X_0 \subset X \subset X_1$ with $X = [X_0, X_1]_{s,p;K}$ we have

$$\|u\|_X \leq c_{s,p} \|u\|_{X_0}^{1-s} \|u\|_{X_1}^{s} \qquad \forall u \in X^1 \tag{5}$$

with

$$c_{s,p} = \begin{cases} [\frac{1}{ps(1-s)}]^{1/p} & , 1 \leq p < \infty \\ 1 & , p = \infty. \end{cases}$$

Convexity inequality - Proof (I)

Proof for $\boxed{0 < s < 1, 1 \leq p < \infty}$:

$$
\begin{aligned}
\|u\|_X^p &= \|u\|^p_{[X_0,X_1]_{s,p;K}} \\
&= \int_0^\infty t^{-ps} K^p(t;u) \frac{dt}{t} \\
&= \int_0^\alpha t^{-ps} K^p(t;u) \frac{dt}{t} + \int_\alpha^\infty t^{-ps} K^p(t;u) \frac{dt}{t} \\
&\leq \int_0^\alpha t^{-ps} t^p \|u\|^p_{X_1} \frac{dt}{t} + \int_\alpha^\infty t^{-ps} \|u\|^p_{X_0} \frac{dt}{t} \\
&= \int_0^\alpha t^{-ps+p-1} dt \|u\|^p_{X_1} + \int_\alpha^\infty t^{-ps-1} dt \|u\|^p_{X_0} \\
&= \frac{1}{p(1-s)} \alpha^{p(1-s)} \|u\|^p_{X_1} + \frac{1}{ps} \alpha^{-ps} \|u\|^p_{X_0}
\end{aligned}
$$

Convexity inequality - Proof (II)

Choose α such that

$$\alpha^{p(1-s)}\|u\|_{X_1}^p = \alpha^{-ps}\|u\|_{X_0}^p.$$

Hence, $\alpha = \frac{\|u\|_{X_0}}{\|u\|_{X_1}}$. Inserting α into the inequality before leads to

$$\begin{aligned}
\|u\|_X^p \quad &\leq \frac{1}{p(1-s)}\left(\frac{\|u\|_{X_0}}{\|u\|_{X_1}}\right)^{p(1-s)}\|u\|_{X_1}^p + \frac{1}{ps}\left(\frac{\|u\|_{X_0}}{\|u\|_{X_1}}\right)^{-ps}\|u\|_{X_0}^p \\
&= \frac{1}{p(1-s)}\|u\|_{X_0}^{p(1-s)}\|u\|_{X_1}^{ps} + \frac{1}{ps}\|u\|_{X_0}^{p(1-s)}\|u\|_{X_1}^{ps} \\
&= \frac{1}{ps(1-s)}\|u\|_{X_0}^{p(1-s)}\|u\|_{X_1}^{ps}.
\end{aligned}$$

$$\Longrightarrow \|u\|_X \leq \left(\tfrac{1}{ps(1-s)}\right)^{\frac{1}{p}}\|u\|_{X_0}^{1-s}\|u\|_{X_1}^{s} \quad \forall u \in X_1. \qquad \blacksquare$$

Bernstein's (inverse) inequalities

Lemma (Bernstein's inequalities)

There exists a constant $c_s > 0$ such that

$$\|v\|_s \leq c_s h_K^{-s} \|v\|_0 \quad \forall v \in M_K \tag{6}$$

for $0 \leq s \leq \gamma < \frac{3}{2}$.

Proof

Summarizing

1. Jackson's (approximation) inequalities

$$\|(I - Q_K)u\|_0 \leq c_2^s h_K^{2s} \|u\|_{2s} \quad \forall u \in H^{2s}(\Omega)$$

   and $0 \leq s \leq 1$

2. Bernstein's (inverse) inequalities

$$\|v\|_s \leq c_s h_K^{-s} \|v\|_0 \quad \forall v \in M_K$$

   and $0 \leq s \leq \gamma < \frac{3}{2}$

Exercise

Let $\Omega = (0,1)$, $v(x)$ is given by

$$v(x) = \begin{cases} 0 & x \in (0, \frac{1}{2}] \\ 1 & x \in (\frac{1}{2}, 1). \end{cases}$$

Show: $v \notin H^{\frac{1}{2}}(0,1)$, but $v \in H^{\frac{1}{2}-\epsilon}(0,1)$.