Interpolation Spaces - The J-method

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Intermediate Spaces

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Algebraic sum $X_0 + X_1 = \{u = u_0 + u_1 : u_0 \in X_0, u_1 \in X_1\}$:
Banach space wrt the norm

$$\|u\|_{X_0 + X_1} = \inf_{u = u_0 + u_1} \{\|u_0\|_{X_0} + \|u_1\|_{X_1}\}$$

$X_0 \cap X_1 \to X_i \to X_0 + X_1$ for $i = 0, 1$

Definition

A Banach space X is called *intermediate* between X_0 and X_1 if there exist the embeddings

$$X_0 \cap X_1 \to X \to X_0 + X_1.$$

The J and K norms

Equivalent norms to $\|\cdot\|_{X_0 \cap X_1}$ and $\|\cdot\|_{X_0+X_1}$:

$$J(t;u) = \max\{\|u\|_{X_0}, t\|u\|_{X_1}\}$$

$$K(t;u) = \inf_{u=u_0+u_1}\{\|u_0\|_{X_0} + t\|u_1\|_{X_1}\}$$

for each $t > 0$ fixed are continuous and monotonically increasing functions with

$$\min\{1,t\}\|u\|_{X_0 \cap X_1} \le J(t;u) \le \max\{1,t\}\|u\|_{X_0 \cap X_1} \tag{1}$$

$$\min\{1,t\}\|u\|_{X_0+X_1} \le K(t;u) \le \max\{1,t\}\|u\|_{X_0+X_1} \tag{2}$$

$$K(t;u) \le \min\{1,\frac{t}{s}\}J(s;u) \quad (\text{for } u \in X_0 \cap X_1, t,s > 0) \tag{3}$$

$J(t,u)$ is a convex function of t

The K-method

If $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$ we denote by

$$(X_0, X_1)_{\theta,q;K}$$

the space of all $u \in X_0 + X_1$ such that the function

$$t \to t^{-\theta} K(t; u)$$

belongs to L^q_*.

Theorem

If and only if either $1 < q \leq \infty$, $0 < \theta < 1$ *or* $q = 1$, $0 \leq \theta \leq 1$, *then* $(X_0, X_1)_{\theta,q;K}$ *is a nontrivial Banach space with norm*

$$\|u\|_{\theta,q;K} = \begin{cases} \left(\int_0^\infty [t^{-\theta} K(t;u)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty \\ \text{ess sup}_{0<t<\infty}\{t^{-\theta} K(t;u)\} & \text{if } q = \infty. \end{cases}$$

Furthermore,

$$\|u\|_{X_0+X_1} \leq \frac{\|u\|_{\theta,q;K}}{\|t^{-\theta}\min\{1,t\}\|_{L^q_*}} \leq \|u\|_{X_0 \cap X_1} \tag{4}$$

so that

$$X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;K} \to X_0 + X_1.$$

Otherwise $(X_0, X_1)_{\theta,q;K} = \{0\}$.

A Discrete Version of the K-method

Theorem

For each integer i let

$$K_i(u) = K(2^i; u).$$

Then $u \in (X_0, X_1)_{\theta,q;K}$ if and only if the sequence

$$\{2^{-i\theta} K_i(u)\}_{i=-\infty}^{\infty}$$

belongs to the space ℓ^q. Moreover, the ℓ^q-norm of that sequence is equivalent to $\|u\|_{\theta,q;K}$.

Proof motivation (for $1 \leq q < \infty$):

$$\|u\|_{\theta,q;K}^q = \int_0^\infty (t^{-\theta} K(t;u))^q \frac{dt}{t} = \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} (t^{-\theta} K(t;u))^q \frac{dt}{t}$$

The J-method

If $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$ we denote by

$$(X_0, X_1)_{\theta,q;J}$$

the space of all $u \in X_0 + X_1$ such that

$$u = \int_0^\infty f(t) \frac{dt}{t}$$

for some

$$f \in L^1(0, \infty; dt/t, X_0 + X_1)$$

having values in $X_0 \cap X_1$ and such that the real-valued function

$$t \to t^{-\theta} J(t; f)$$

belongs to L^q_*.

Theorem

If either $1 < q \leq \infty$, $0 < \theta < 1$ *or* $q = 1$, $0 \leq \theta \leq 1$, *then* $(X_0, X_1)_{\theta,q;J}$ *is a nontrivial Banach space with norm*

$$\|u\|_{\theta,q;J} = \inf_{f \in S(u)} \|t^{-\theta} J(t; f(t))\|_{L^q_*},$$

where $S(u) = \{f \in L^1(0,\infty; dt/t, X_0 + X_1) : u = \int_0^\infty f(t) \frac{dt}{t}\}$. *Furthermore,*

$$\|u\|_{X_0+X_1} \leq \|t^{-\theta} \min\{1,t\}\|_{L^{q'}_*} \|u\|_{\theta,q;J} \leq \|u\|_{X_0 \cap X_1} \tag{5}$$

so that

$$X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;J} \to X_0 + X_1.$$

Theorem - Proof (I)

Proof of $\|u\|_{X_0+X_1} \leq \|t^{-\theta}\min\{1,t\}\|_{L_*^{q'}}\|u\|_{\theta,q;J}$:

Let $f \in S(u)$. By (2) and (3) with $t = 1$ and $s = \tau$ we have

$$\|f(\tau)\|_{X_0+X_1} \leq K(1, f(\tau)) \leq \min\{1, \frac{1}{\tau}\}J(\tau, f(\tau)). \tag{6}$$

By the triangle inequality ($\|\cdot\|_{X_0+X_1}$ is a norm), by (6) and if $\frac{1}{q} + \frac{1}{q'} = 1$, by Hölder's inequality it follows

$$\begin{aligned}\|u\|_{X_0+X_1} \leq \int_0^\infty \|f(\tau)\|_{X_0+X_1}\frac{d\tau}{\tau} &\leq \int_0^\infty \min\{1, \frac{1}{\tau}\}J(\tau, f(\tau))\frac{d\tau}{\tau} \\ &= \int_0^\infty \min\{1, \frac{1}{\tau}\}\tau^\theta\tau^{-\theta}J(\tau, f(\tau))\frac{d\tau}{\tau} \\ &\leq \|\tau^\theta \min\{1, \frac{1}{\tau}\}\|_{L_*^{q'}}\|t^{-\theta}J(t; f(t))\|_{L_*^q}.\end{aligned}$$

Theorem - Proof (II)

The inequality holds for all $f \in S(u)$ (hence also for the infimum of them). Together with setting $\tau = \frac{1}{t}$ in $\|\tau^{\theta} \min\{1, \frac{1}{\tau}\}\|_{L_*^{q'}}$, which is finite, if θ and q satisfy the conditions of the theorem, we have that

$$\|u\|_{X_0+X_1} \leq \|t^{-\theta} \min\{1, t\}\|_{L_*^{q'}} \|u\|_{\theta,q;J}.$$

Proof of $\|t^{-\theta} \min\{1, t\}\|_{L_*^{q'}} \|u\|_{\theta,q;J} \leq \|u\|_{X_0 \cap X_1}$:

Let $u \in X_0 \cap X_1$. Let $\phi(t) \geq 0$ satisfy $\|t^{-\theta}\phi(t)\|_{L_*^q} = 1$. By Hölder's inequality it follows

$$\int_0^{\infty} \phi(\tau) \min\{1, \frac{1}{\tau}\} \frac{d\tau}{\tau} = \int_0^{\infty} \phi(\tau)\tau^{\theta}\tau^{-\theta} \min\{1, \frac{1}{\tau}\} \frac{d\tau}{\tau}$$
$$\leq \|\tau^{-\theta}\phi(\tau)\|_{L_*^q} \|\tau^{\theta} \min\{1, \frac{1}{\tau}\}\|_{L_*^{q'}} = \|\tau^{\theta} \min\{1, \frac{1}{\tau}\}\|_{L_*^{q'}} < \infty.$$

Theorem - Proof (III)

If

$$f(t) = \frac{\phi(t)\min\{1,\frac{1}{t}\}}{\int_0^\infty \phi(\tau)\min\{1,\frac{1}{\tau}\}\frac{d\tau}{\tau}}u,$$

then $f \in S(u)$ and

$$\begin{aligned} J(t;f(t)) &= \frac{\phi(t)\min\{1,\frac{1}{t}\}}{\int_0^\infty \phi(\tau)\min\{1,\frac{1}{\tau}\}\frac{d\tau}{\tau}}J(t;u) \\ &\leq \frac{\phi(t)}{\int_0^\infty \phi(\tau)\min\{1,\frac{1}{\tau}\}\frac{d\tau}{\tau}}\|u\|_{X_0\cap X_1}, \end{aligned}$$

which follows from (1) since $\max\{1,t\} = (\min\{1,\frac{1}{t}\})^{-1}$. Hence,

$$\left(\int_0^\infty \phi(\tau)\min\{1,\frac{1}{\tau}\}\frac{d\tau}{\tau}\right) J(t;f(t)) \leq \phi(t)\|u\|_{X_0\cap X_1}. \qquad (7)$$

Theorem - Proof (IV)

By the definition of $\|\cdot\|_{\theta,q;J}$ and by (7) we have

$$\left(\int_0^\infty \phi(\tau)\min\{1,\frac{1}{\tau}\}\frac{d\tau}{\tau}\right)\|u\|_{\theta,q;J}$$

$$\leq \left(\int_0^\infty \phi(\tau)\min\{1,\frac{1}{\tau}\}\frac{d\tau}{\tau}\right)\|t^{-\theta}J(t;f(t))\|_{L^q_*}$$

$$\leq_{(7)} \left(\int_0^\infty (t^{-\theta}\phi(t)\|u\|_{X_0\cap X_1})^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty (t^{-\theta}\phi(t))^q\frac{dt}{t}\right)^{\frac{1}{q}}\|u\|_{X_0\cap X_1} = \|u\|_{X_0\cap X_1}$$

because $\|t^{-\theta}\phi(t)\|_{L^q_*} = 1$.

Theorem - Proof (V)

By the converse to Hölder's inequality, we get

$$\sup\{\int_0^\infty \phi(\tau)\min\{1,\frac{1}{\tau}\}\frac{d\tau}{\tau} : \|\tau^{-\theta}\phi(\tau)\|_{L_*^q} = 1\}$$
$$= \|\tau^{\theta}\min\{1,\frac{1}{\tau}\}\|_{L_*^{q'}} =_{\tau=\frac{1}{t}} \|t^{-\theta}\min\{1,t\}\|_{L_*^{q'}}$$

and we have proven

$$\|t^{-\theta}\min\{1,t\}\|_{L_*^{q'}}\|u\|_{\theta,q;J} \leq \|u\|_{X_0\cap X_1}. \qquad \blacksquare$$

A Discrete Version of the J-method

Theorem

An element u of $X_0 + X_1$ belongs to $(X_0, X_1)_{\theta,q;J}$ if and only if $u = \sum_{i=-\infty}^{\infty} u_i$ where the series converges in $X_0 + X_1$ and the sequence

$$\{2^{-i\theta} J(2^i; u_i)\}_{i=-\infty}^{\infty}$$

belongs to ℓ^q. In this case

$$\inf\{\|\{2^{-i\theta} J(2^i; u_i)\}\|_{\ell^q} : u = \sum_{i=-\infty}^{\infty} u_i\}$$

is a norm on $(X_0, X_1)_{\theta,q;J}$ equivalent to $\|u\|_{\theta,q;J}$.

Equivalence Theorem

Theorem

If $0 < \theta < 1$ *and* $1 \leq q \leq \infty$*, then*

$$(X_0, X_1)_{\theta,q;J} \to (X_0, X_1)_{\theta,q;K}$$

and

$$(X_0, X_1)_{\theta,q;K} \to (X_0, X_1)_{\theta,q;J}.$$

Therefore,

$$(X_0, X_1)_{\theta,q;J} = (X_0, X_1)_{\theta,q;K},$$

i.e. the two corresponding norms are equivalent.

Equivalence Theorem - Proof (I)

Proof of $(X_0, X_1)_{\theta,q;J} \to (X_0, X_1)_{\theta,q;K}$:

Let $u = \int_0^\infty f(s)\frac{ds}{s} \in (X_0, X_1)_{\theta,q;J}$.
Since $K(t, \cdot)$ is a norm on $X_0 + X_1$, we have that

$$\begin{aligned}
t^{-\theta}K(t; u) \quad & \leq t^{-\theta}\int_0^\infty K(t; f(s))\frac{ds}{s} \\
& \leq_{(3)} t^{-\theta}\int_0^\infty \min\{1, \frac{t}{s}\}J(s; f(s))\frac{ds}{s} \\
& = \int_0^\infty \left(\frac{t}{s}\right)^{-\theta} \min\{1, \frac{t}{s}\}s^{-\theta}J(s; f(s))\frac{ds}{s} \\
& = [t^{-\theta}\min\{1, t\}] * [t^{-\theta}J(t; f(t))].
\end{aligned}$$

Equivalence Theorem - Proof (II)

By Young's inequality for convolution, we obtain

$$\begin{aligned} \|u\|_{\theta,q;K} &= \|t^{-\theta} K(t;u)\|_{L^q_*} \\ &\leq \|t^{-\theta}\min\{1,t\}\|_{L^1_*} \|t^{-\theta} J(t;f(t))\|_{L^q_*} \\ &= C_{\theta,q}\|u\|_{\theta,q;J}. \end{aligned}$$

Proof of $(X_0, X_1)_{\theta,q;K} \to (X_0, X_1)_{\theta,q;J}$: using discrete versions of the J and K methods.
Let $u \in (X_0, X_1)_{\theta,q;K}$. By the definition of $K(t;u)$, for each integer i there exist $v_i \in X_0$ and $w_i \in X_1$ such that

$$u = v_i + w_i \quad \text{and} \quad \|v_i\|_{X_0} + 2^i \|w_i\|_{X_1} \leq 2K(2^i; u).$$

Equivalence Theorem - Proof (III)

Then $\{2^{-i\theta}\|v_i\|_{X_0}\}$ and $\{2^{-i\theta}2^i\|w_i\|_{X_1}\}$ belong to ℓ^q and have ℓ^q-norm bounded by a constant times $\|u\|_{\theta,q;K}$.
We have

$$0 = u - u = (v_{i+1} + w_{i+1}) - (v_i + w_i) = (v_{i+1} - v_i) + (w_{i+1} - w_i)$$

$$\Rightarrow v_{i+1} - v_i = w_i - w_{i+1}.$$

Let $u_i = v_{i+1} - v_i$ for all integers i. Therefore,

$$u_i = v_{i+1} - v_i = w_i - w_{i+1},$$

and $\{2^{-i\theta}\|u_i\|_{X_0}\} \in \ell^q$ and $\{2^{-i\theta}2^i\|u_i\|_{X_1}\} \in \ell^q$ have ℓ^q-norm bounded by a constant times $\|u\|_{\theta,q;K}$. So, $\{2^{-i\theta}J(2^i;u_i)\} \in \ell^q$ and has ℓ^q-norm bounded by a constant times $\|u\|_{\theta,q;K}$.

Equivalence Theorem - Proof (IV)

Since $\ell^q \subset \ell^\infty$, $\{2^{j(1-\theta)}\|w_j\|_{X_1}\}$ is bounded even though $2^{j(1-\theta)} \to \infty$ as $j \to \infty$. So, $\|w_j\|_{X_1} \to 0$ as $j \to \infty$. Since

$$\sum_{i=0}^{j} u_i = w_0 - w_{j+1},$$

the half series $\sum_{i=0}^{\infty} u_i$ converges to w_0 in X_1 and hence in $X_0 + X_1$. Similary, the half series $\sum_{i=-\infty}^{-1} u_i$ converges to v_0 in X_0 and hence in $X_0 + X_1$. Altogether, the full series $\sum_{i=-\infty}^{\infty} u_i$ converges to

$$v_0 + w_0 = u \quad \text{in} \quad X_0 + X_1,$$

and we have

$$\|u\|_{\theta,q;J} \leq c_{discrJ}\|\{2^{-i\theta}J(2^i;u_i)\}\|_{\ell^q} \leq C\|u\|_{\theta,q;K}. \qquad \blacksquare$$

Interpolation Spaces

Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs of Banach spaces, let operator $T : X_0 + X_1 \to Y_0 + Y_1$ be bounded and linear with

$$\|Tu_i\|_{Y_i} \le M_i \|u_i\|_{X_i}, \quad \forall u_i \in X_i \quad (i = 0, 1).$$

If X and Y are intermediate spaces for P and Q and if every such linear operator T maps X into Y with norm M satisfying

$$M \le C M_0^{1-\theta} M_1^{\theta}, \tag{8}$$

where constant $C \ge 1$ is independent of T and $0 \le \theta \le 1$, then X and Y are called *interpolation spaces of type* θ for P and Q. The interpolation spaces X and Y are *exact* if (6) holds with $C = 1$. If $X_0 = Y_0$, $X_1 = Y_1$, $X = Y$ and $T = I$, then $C = 1$ for all $0 \le \theta \le 1$, so no smaller C is possible in (6).

An Exact Interpolation Theorem

Theorem

Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs of Banach spaces.

(a) *If either $0 < \theta < 1$, $1 \leq q \leq \infty$ or $0 \leq \theta \leq 1, q = \infty$, then the intermediate spaces $(X_0, X_1)_{\theta,q;K}$ and $(Y_0, Y_1)_{\theta,q;K}$ are exact interpolation spaces of type θ for P and Q.*

(b) *If either $0 < \theta < 1$, $1 < q \leq \infty$ or $0 \leq \theta \leq 1, q = 1$, then the intermediate spaces $(X_0, X_1)_{\theta,q;J}$ and $(Y_0, Y_1)_{\theta,q;J}$ are exact interpolation spaces of type θ for P and Q.*