MULTILEVEL REPRESENTATIONS, SPACE INTERPOLATION AND PRECONDITIONING
SEMINAR ON NUMERICAL ANALYSIS - WS 2010/11

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Abstract. The topic of this report is taken from our presentations in the seminar *Interpolation Spaces and Applications in Numerical Analysis* at the *Institute of Computational Mathematics, JKU Linz.* The report deals with preconditioning, Jackson's and Bernstein's inequalities in Sobolev-Slobodecki spaces and multilevel representation of the H^s-norm. The subject matter is taken from lectures of Bramble. Moreover, the reader is referred to have a look at [2] and [4] for more details.

1. Preconditioning

1.1. **Existence and uniqueness of a variational problem.** We consider the variational problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u, \varphi) = \langle f, \varphi \rangle \qquad \forall \varphi \in H_0^1(\Omega) =: V_0 \tag{1}$$

for a given $f \in (H_0^1(\Omega))^* = H^{-1}(\Omega)$. In the following, we assume that the bilinear form

$$a(\cdot, \cdot) : V_0 \times V_0 \to \mathbb{R}$$

is symmetric, V_0-elliptic and V_0-bounded, which guarantees the existence of a unique solution of the variational problem (by the Lax-Milgram theorem).

1.2. **Galerkin approximation.** Let $M_h \subset H_0^1(\Omega)$ be a finite dimensional subspace. Then we have the following Galerkin scheme: Find $u_h \in M_h$ such that

$$a(u_h, \varphi_h) = \langle f, \varphi_h \rangle \qquad \forall \varphi_h \in M_h =: M.$$

Let $\{\varphi_i\}_{i=\overline{1,\dim M_h}}$ be a basis for M_h. Inserting the representation

$$u_h = \sum_{i=1}^{\dim M_h} u_i \varphi_i$$

into the equation above leads to

$$\begin{aligned} \sum_{i=1}^{\dim M_h} u_i a(\varphi_i, \varphi_j) &= \langle f, \varphi_j \rangle, \qquad j = \overline{1, \dim M_h} \\ \underline{\underline{A}}_h \underline{u}_h &= \underline{f}_h, \end{aligned}$$

where $\underline{\underline{A}}_h$ has the entries $a(\varphi_i, \varphi_j)$, $\underline{u}_h$ the entries u_i and $\underline{f}_h$ the entries $\langle f, \varphi_j \rangle$.

1.3. **Spectral condition number and preconditioning.** Due to our assumptions on the bilinear form $a(\cdot, \cdot)$, our matrix $\underline{\underline{A}}_h$ is symmetric positive definite and we get the following (basis dependent) condition number by choosing a nodal FE-basis:

$$\kappa(\underline{\underline{A}}_h) = cond_2(\underline{\underline{A}}_h) = \frac{\lambda_{\max}(\underline{\underline{A}}_h)}{\lambda_{\min}(\underline{\underline{A}}_h)} = O(h^{-2}).$$

Other approach: Define the symmetric positive definite (SPD) operator $A_h : M_h \to M_h^*$ by

$$(A_h u_h, \varphi_h)_{L_2(\Omega)} = a(u_h, \varphi_h) \qquad \forall u_h, \varphi_h \in M_h$$

and the (basis independent) operator condition number is

$$\kappa(A_h) = \frac{\lambda_{\max}(A_h)}{\lambda_{\min}(A_h)} = O(h^{-2}).$$

1.4. **Properties of a preconditioner.** Let $B_h^{-1} : M_h^* \to M_h$ be another SPD operator. A "good preconditioner" should have the following properties:

- the action of B_h^{-1} on M_h^* is "cheap"
- the condition number $\kappa(B_h^{-1}A_h) << \kappa(A_h)$

Two extreme cases:

- $B_h^{-1} = I_h$ - cheap, but the second property is not valid
- $B_h^{-1} = A_h^{-1}$ - we have that $\kappa(B_h^{-1}A_h) = \kappa(I_h) = 1$, but that's not cheap

In the following, we will use the notation: $A_h = A, B_h = B$.

Proposition 1. *Suppose there exist constants $c_1, c_2 > 0$ with*

$$c_1(Bv, v) \le (Av, v) \le c_2(Bv, v) \qquad \forall v \in M. \tag{2}$$

Then $\kappa(B^{-1}A) \le c_2/c_1$.

2. Preliminaries

2.1. **Geometry and Mesh.** Bounded domain $\Omega \subset \mathbb{R}^d$ (here $d = 2$)

Family of meshes $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ with meshsize h_k

- Triangulation with triangles
- global quasi uniform (i.e $\frac{\max_k h_k}{\min_k h_k} \le c$)
- nested dyadic refinement

$$c_1 2^{-k} \le h_k \le c_2 2^{-k} \tag{3}$$

Familiy of nested finite element spaces $\{M_k\}_{k\in\mathbb{N}}$

$$M_1 \subset M_2 \subset M_3 \subset \ldots \subset M_k \subset \ldots \subset H^1(\Omega) =: V$$

where each space M_k corresponds to $S^1_{h_k}$ (set of continuous, piecewise linear functions).

3. Jackson's and Bernstein's inequalities in Sobolev-Slobodecki spaces

Let $\{Q_k\}_{k\in\mathbb{N}}$ be a family of L_2-projections, where $Q_k : L_2(\Omega) \to M_k$ and Q_k is defined by

$$(Q_k u, \varphi_k)_0 = (u, \varphi_k)_0, \quad \forall \varphi_k \in M_k,\ \forall u \in L_2(\Omega). \tag{4}$$

Additionally, it should be valid

$$(Q_k u, \varphi_k)_0 = \langle u, \varphi_k \rangle, \quad \forall \varphi_k \in M_k,\ \forall u \in V^*,$$

where we have the notation $\|\cdot\|_0 = \|\cdot\|_{L_2(\Omega)}$ and $(\cdot,\cdot)_0 = (\cdot,\cdot)_{L_2(\Omega)}$.

Lemma 1. *For the sequence $\{Q_i\}_{i\in\mathbb{N}}$ of L_2-projection operators we have the following properties:*

1. $Q_k u \in M_k$
2. $Q_k Q_j = Q_j Q_k = Q_{\min\{k,j\}}$
3. $\|Q_k\| = 1$

Proof. We have to distinguish three cases: For $u_j \in M_j$, we have $Q_j v_j = v_j \in M_j$ and therefore $Q_j Q_j v = Q_j v$ for all $v \in L_2(\Omega)$. In the case $j < k$ we have $M_j \subset M_k$. Then $Q_j v \in M_j \subset M_k$ and $Q_k Q_j v = Q_j v$. In the case $j < k$ we have

$$(Q_k v, v_j)_0 = (v, v_j)_0, \quad \forall v_j \in M_j$$

and therefore

$$(Q_k Q_j v, v_k)_0 = (Q_j v, v_k)_0 = (v, v_k)_0 = (Q_k v, v_k)_0, \quad \forall v_k \in M_k \subset M_j.$$

The last property follows from

$$\|Q_k u\|_0 = \|Q_k^2 u\|_0 \leq \|Q_k\| \|Q_k u\|_0.$$

□

Lemma 2 (Jackson's (approximation) inequalities)**.** *We have*

(1) $\|(I - Q_k)u\|_0 \leq ch_k^2 \|u\|_2, \quad \forall u \in H^2(\Omega)$
(2) $\|(I - Q_k)u\|_0 \leq 1 \cdot \|u\|_0, \quad \forall u \in L_2(\Omega)$
(3) $\|(I - Q_k)u\|_0 \leq c^s h_k^{2s} \|u\|_{2s}, \quad \forall u \in H^{2s}(\Omega)$

with $H^{2s}(\Omega) = [L_2(\Omega), H^2(\Omega)]_s$*,* $0 < s < 1$*.*

Proof. (1): We have

$$\begin{aligned}\|(I - Q_k)u\|_0^2 &= ((I - Q_k)u, (I - Q_k)u)_0 = ((I - Q_k)u, u - Q_k u)_0 \\ &= ((I - Q_k)u, u - \varphi_k)_0 \leq \|(I - Q_k)u\|_0 \|u - \varphi_k\|_0\end{aligned}$$

for all $\varphi_k \in M_k$, where we have used (4). Hence

$$\|(I - Q_k)u\|_0 \leq \inf_{\varphi_k \in M_k} \|u - \varphi_k\|_0.$$

With φ_k being the interpolant of u, we obtain

$$\inf_{\varphi_k \in M_k} \|u - \varphi_k\|_0 \leq ch_k^2 \|u\|_2$$

for all $u \in H^2(\Omega)$, which proves statement (1).
(2): By (1) and taking $\varphi_k = 0 \in M_k$, we get

$$\|(I - Q_k)u\|_0 \leq \inf_{\varphi_k \in M_k} \|u - \varphi_k\|_0 \leq \|u - 0\|_0 = 1 \cdot \|u\|_0$$

for all $u \in L_2(\Omega)$.
(3): By applying the space interpolation theorem (for more details, see [1]) with (1) and (2), we obtain

$$\|(I - Q_k)u\|_0 \leq 1^{1-s}(ch_k^2)^s \|u\|_{2s} = c^s h_k^{2s} \|u\|_{2s}$$

for all $u \in H^{2s}(\Omega)$.
Altogether, we have proven

$$\|(I - Q_k)u\|_0 \leq c^s h_k^{2s} \|u\|_{2s}$$

for all $u \in H^{2s}(\Omega)$ with $0 \leq s \leq 1$. □

As next, we want to prove Bernstein's inequalities. For that, we need an important tool: the convexity inequality.

Lemma 3 (Convexity inequality)**.** *For* $X_0 \subset X \subset X_1$ *with* $X = [X_0, X_1]_{s,p;K}$ *we have*

$$\|u\|_X \leq c_{s,p} \|u\|_{X_0}^{1-s} \|u\|_{X_1}^s, \quad \forall u \in X^1$$

with

$$c_{s,p} = \begin{cases} \left(\frac{1}{ps(1-s)}\right)^{\frac{1}{p}} & , 1 \leq p < \infty \\ 1 & , p = \infty. \end{cases}$$

Proof. Proof for $0 < s < 1, 1 \le p < \infty$:

$$
\begin{aligned}
\|u\|_X^p \quad &= \|u\|^p_{[X_0,X_1]_{s,p;K}} \\
&= \int_0^\infty t^{-ps} K^p(t;u) \frac{dt}{t} \\
&= \int_0^\alpha t^{-ps} K^p(t;u) \frac{dt}{t} + \int_\alpha^\infty t^{-ps} K^p(t;u) \frac{dt}{t} \\
&\le \int_0^\alpha t^{-ps} t^p \|u\|^p_{X_1} \frac{dt}{t} + \int_\alpha^\infty t^{-ps} \|u\|^p_{X_0} \frac{dt}{t} \\
&= \int_0^\alpha t^{-ps+p-1} dt \|u\|^p_{X_1} + \int_\alpha^\infty t^{-ps-1} dt \|u\|^p_{X_0} \\
&= \frac{1}{p(1-s)} \alpha^{p(1-s)} \|u\|^p_{X_1} + \frac{1}{ps} \alpha^{-ps} \|u\|^p_{X_0}
\end{aligned}
$$

Choose α such that

$$\alpha^{p(1-s)} \|u\|^p_{X_1} = \alpha^{-ps} \|u\|^p_{X_0}.$$

Hence, $\alpha = \frac{\|u\|_{X_0}}{\|u\|_{X_1}}$. Inserting α into the inequality before leads to

$$
\begin{aligned}
\|u\|_X^p \quad &\le \frac{1}{p(1-s)} \left(\frac{\|u\|_{X_0}}{\|u\|_{X_1}} \right)^{p(1-s)} \|u\|^p_{X_1} + \frac{1}{ps} \left(\frac{\|u\|_{X_0}}{\|u\|_{X_1}} \right)^{-ps} \|u\|^p_{X_0} \\
&= \frac{1}{p(1-s)} \|u\|^{p(1-s)}_{X_0} \|u\|^{ps}_{X_1} + \frac{1}{ps} \|u\|^{p(1-s)}_{X_0} \|u\|^{ps}_{X_1} \\
&= \frac{1}{ps(1-s)} \|u\|^{p(1-s)}_{X_0} \|u\|^{ps}_{X_1}.
\end{aligned}
$$

$\Longrightarrow \|u\|_X \le \left(\frac{1}{ps(1-s)} \right)^{\frac{1}{p}} \|u\|^{1-s}_{X_0} \|u\|^s_{X_1} \quad \forall u \in X_1.$
The proof for $p = \infty$ is analogous and is left to the reader. $\square$

For proving Bernstein's inequalities, we need the logarithmic convexity inequality with $X_0 = L_2(\Omega)$ and $X_1 = H^1(\Omega)$, i.e.

Lemma 4 (Logarithmic convexity inequality)**.**

$$\|u\|_s \le c_{s,2} \|u\|_0^{1-s} \|u\|_1^s, \quad \forall u \in H^1(\Omega)$$

with $c_{s,2} = \frac{1}{\sqrt{2s(1-s)}}$.

Here, $H^s(\Omega) = [L_2(\Omega), H^1(\Omega)]_s$ and

$$\|u\|_s^2 = \|u\|^2_{H^s(\Omega)} := \|u\|_1^2 + \sum_{|\alpha|=1} \|D^\alpha u\|^2_\beta,$$

where $s = 1 + \beta$ and $0 < \beta < \frac{1}{2}$. Now, we can prove Bernstein's inequalities.

Lemma 5 (Bernstein's (inverse) inequalities)**.** *There exists a constant $c > 0$ such that*

$$\|u\|_s \le c h_k^{-s} \|u\|_0, \quad \forall u \in M_k$$

for $0 \le s < \frac{3}{2}$.

Proof. Case 1: $0 \le s \le 1$:
For $s = 1$, we use a statement from the lectures of *Numerical Methods for Elliptic Partial Differential Equations*, i.e.

$$\|u\|_1 \le c h_k^{-1} \|u\|_0, \quad \forall u \in M_k.$$

Together with the logarithmic convexity inequality from Lemma 4, we obtain

$$\begin{aligned}\|u\|_s &\leq c_{s,2}\|u\|_0^{1-s}\|u\|_1^s \\ &\leq c_{s,2}\|u\|_0^{1-s}(ch_k^{-1}\|u\|_0)^s \\ &\leq c_{s,2}c^s h_k^{-s}\|u\|_0^{1-s}\|u\|_0^s \\ &= ch_k^{-s}\|u\|_0\end{aligned}$$

with a constant $c > 0$.
Case 2: $1 \leq s < 1+\beta$ with $0 < \beta < \frac{1}{2}$:
It is to show that we have for all piecewise constant functions over each triangle in $\mathcal{T}_k$:

$$\|w\|_\beta \leq ch_k^{-\beta}\|w\|_0.$$

The norm is defined by

$$\|w\|_\beta = \int_0^\infty t^{-2\beta-1}K^2(t;w)dt$$

with

$$K^2(t;w) = \inf_{v\in H^1(\Omega)}(\|w-v\|_0^2 + t^2\|v\|_1^2)$$

(for more information, see Seminar 01 and Seminar 02). Now, we have to consider the two limit cases, i.e. t around zero and t around infinity. For that, we split the integral into two integrals.
(a) t around infinity:

$$\begin{aligned}\int_{h_k}^\infty t^{-2\beta-1}K^2(t;w)dt &= \int_{h_k}^\infty t^{-2\beta-1}\inf_{v\in H^1(\Omega)}(\|w-v\|_0^2 + t^2\|v\|_1^2)dt \\ &\leq \int_{h_k}^\infty t^{-2\beta-1}\|w\|_0^2 dt\end{aligned}$$

by choosing $v = 0$ in the last step. Hence,

$$\int_{h_k}^\infty t^{-2\beta-1}K^2(t;w)dt \leq \int_{h_k}^\infty t^{-2\beta-1}dt\|w\|_0^2 = \frac{1}{2\beta}h_k^{-2\beta}\|w\|_0^2.$$

So, $\int_{h_k}^\infty t^{-2\beta-1}K^2(t;w)dt \leq ch_k^{-2\beta}\|w\|_0^2$.
(b) t around zero:
We have a look at the integral

$$\int_0^{h_k} t^{-2\beta-1}K^2(t;w)dt.$$

The aim is to find a proper $v \in H^1(\Omega)$ such that $K^2(t;w)$ is around 0, for t around 0. For that, we take a fixed triangle τ_i and define, for $t \leq h_k$, a smooth function ϕ_i on Ω as follows:

$$\phi_i(x) = \begin{cases} 0 & , x \in \tau_i \\ 1 & , dist(x, \partial\tau_i) \geq t. \end{cases}$$

Then, $|\nabla\phi_i| \leq ct^{-1}$ with a constant $c > 0$. We take

$$v = \sum_i \phi_i w,$$

where w is piecewise constant and $v \in H^1$. Then, we have

$$\begin{aligned}
\|w-v\|^2_{L_2(\tau_i)} &= \int_{\tau_i} |w-v|^2 dx = \int_{\tau_i} |w-\phi_i w|^2 dx \\
&\leq \int_{\tau_i} |1-\phi_i|^2 |w|^2 dx \leq \int_{\tau_i} 1 dx \|w\|^2_{L_\infty(\tau_i)} \\
&= |\tau_i| \|w\|^2_{L_\infty(\tau_i)} \leq c h_k t \|w\|^2_{L_\infty(\tau_i)}
\end{aligned}$$

and

$$\begin{aligned}
\|v\|^2_{H^1(\tau_i)} &= \|v\|^2_{L_2(\tau_i)} + \|\nabla v\|^2_{L_2(\tau_i)} \\
&= \|\phi_i w\|^2_{L_2(\tau_i)} + \|\nabla \phi_i w\|^2_{L_2(\tau_i)} \\
&\leq 1 \cdot \|w\|^2_{L_2(\tau_i)} + \int_{\tau_i} c t^{-2} dx \|w\|^2_{L_\infty(\tau_i)} \\
&\leq \|w\|^2_{L_2(\tau_i)} + c h_k t^{-1} \|w\|^2_{L_\infty(\tau_i)}.
\end{aligned}$$

So, we have

$$\begin{aligned}
K^2(t;w) &= \inf_{v \in H^1(\Omega)} (\|w-v\|^2_0 + t^2 \|v\|^2_1) \\
&\leq \sum_i (\|w-v\|^2_{L_2(\tau_i)} + t^2 \|v\|^2_{H^1(\tau_i)})
\end{aligned}$$

for our choice $v = \sum_i \phi_i w$, and therefore

$$\begin{aligned}
K^2(t;w) &\leq \sum_i c h_k t \|w\|^2_{L_\infty(\tau_i)} + t^2 \sum_i (\|w\|^2_{L_2(\tau_i)} + c h_k t^{-1} \|w\|^2_{L_\infty(\tau_i)}) \\
&= \sum_i c h_k^{-1} h_k^2 t \|w\|^2_{L_\infty(\tau_i)} + t^2 \sum_i (\|w\|^2_{L_2(\tau_i)} + c h_k^{-1} h_k^2 t^{-1} \|w\|^2_{L_\infty(\tau_i)}) \\
&\leq c h_k^{-1} t \|w\|^2_{L_2(\Omega)} + t^2 (\|w\|^2_{L_2(\Omega)} + c h_k^{-1} t^{-1} \|w\|^2_{L_2(\Omega)}) \\
&= c h_k^{-1} t \|w\|^2_{L_2(\Omega)} + t^2 \|w\|^2_{L_2(\Omega)} + c h_k^{-1} t \|w\|^2_{L_2(\Omega)} \\
&\leq c h_k^{-1} t \|w\|^2_{L_2(\Omega)} + h_k^{-1} t \|w\|^2_{L_2(\Omega)} + c h_k^{-1} t \|w\|^2_{L_2(\Omega)}
\end{aligned}$$

because $t \leq h_k$ and $h_k < 1 < h_k^{-1}$. So, $t^2 \leq h_k t \leq h_k^{-1} t$. Altogether, we obtain

$$K^2(t;w) \leq c h_k^{-1} t \|w\|^2_{L_2(\Omega)}$$

and

$$\begin{aligned}
\int_0^{h_k} t^{-2\beta-1} K^2(t;w) dt &\leq \int_0^{h_k} t^{-2\beta-1} c h_k^{-1} t \|w\|^2_0 dt \\
&= c h_k^{-1} \int_0^{h_k} t^{-2\beta} dt \|w\|^2_0 \\
&= c h_k^{-1} \frac{1}{1-2\beta} {h_k}^{1-2\beta} \|w\|^2_0
\end{aligned}$$

and so

$$\int_0^{h_k} t^{-2\beta-1} K^2(t;w) dt \leq c {h_k}^{-2\beta} \|w\|^2_0.$$

Now, we put the two integrals from (a) and (b) together and get

$$\int_{h_k}^{\infty} t^{-2\beta-1} K^2(t;w) dt + \int_0^{h_k} t^{-2\beta-1} K^2(t;w) dt \leq c h_k^{-2\beta} \|w\|^2_0$$

and finally,

$$\|w\|_\beta \leq c h_k^\beta \|w\|_0,$$

with a constant $c > 0$ for all piecewise constant functions w. This proves our statement of case 2 and hence, together with case 1, we have proven the whole lemma. $\square$

Remark 1. *Altogether, we have that*

$$M_k \subset H^{3/2-\epsilon}(\Omega)$$

for any small $\epsilon > 0$*, but* $M_k \not\subset H^{\frac{3}{2}}(\Omega)$*.*

4. Multilevel representation of the H^s-norm and precontioning

We introduce another family of L_2 projectors, that turn out to be more suited since they impose an orthogonal splitting of the space $L_2(\Omega)$.

Definition 1.

$$\begin{aligned} D_1 &:= Q_1 \\ D_k &:= Q_k - Q_{k-1}, \quad k = 2, 3, \dots \end{aligned}$$

Lemma 6. *For the sequence* $\{D_k\}_{k\in\mathbb{N}}$ *of* L_2 *projection operators we have the following properties:*

(1) $D_k u \in M_k$
(2) $D_k D_j = 0$ *for* $k \neq j$ *(orthogonality property).*
(3) $D_k D_k = D_k$ *(projection property).*
(4) D_k *is self-adjoint in* $L^2(\Omega)$*, i.e* $(D_k u, v)_0 = (D_k v, u)_0$*.*

Proof. Since $Q_k u \in M_k$ and $Q_{k-1} u \in M_{k-1} \subset M_k$ we have $D_k u = Q_k u - Q_{k-1} u \in M_k$. The projection property follows from

$$\begin{aligned} D_k^2 = (Q_k - Q_{k-1})^2 &= Q_k^2 - Q_{k-1}Q_k - Q_k Q_{k-1} + Q_{k-1}^2 \\ &= Q_k - Q_{k-1} - Q_{k-1} + Q_{k-1} = Q_k - Q_{k-1} = D_k. \end{aligned}$$

Next without loss of generality we assume $k > j$.

$$\begin{aligned} D_k D_j = (Q_k - Q_{k-1})(Q_j - Q_{j-1}) &= Q_k Q_j - Q_{k-1} Q_j - Q_k Q_{j-1} + Q_{k-1} Q_{j-1} \\ &= Q_j - Q_j - Q_{j-1} + Q_{j-1} = 0 \end{aligned}$$

Finally we have

$$(D_k u, v)_0 = (Q_k u, v)_0 - (Q_{k-1} u, v)_0 = (Q_k v, u)_0 - (Q_{k-1} v, u)_0 = (D_k v, u)_0.$$

$\square$

Hence we can conclude, that D_k is an orthogonal projector wrt $(\cdot,\cdot)_0$. Using this family of orthogonal projectors D_k we can give an orthogonal decomposition of $L_2(\Omega)$.

Corollary 1 (Orthogonal decomposition of $L_2(\Omega)$)**.** *We have*

(1) $L_2(\Omega) = \sum_{k\in\mathbb{N}} \mathcal{O}_k$*, where* $\mathcal{O}_k = \{\varphi : \varphi = D_k u, u \in L_2(\Omega)\}$
(2) $u = \sum_{k\in\mathbb{N}} D_k u$ *in* L_2*-sense.*
(3) $\|u\|_0^2 = \sum_{k\in\mathbb{N}} \|D_k u\|_0^2, \quad \forall u \in L_2(\Omega)$

Proof. First we observe by evaluating a telescoping sum that $\sum_{j=1}^k D_k u = Q_k$. Hence we have

$$\|u - \sum_{j=1}^{k} D_k u\|_0 = \|u - Q_k u\|_0 \stackrel{k\to\infty}{\longrightarrow} 0$$

Furthermore from the orthogonality property we conclude

$$\|u\|_0^2 = \|\sum_{j\in\mathbb{N}} D_j u\|_0^2 = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} (D_j u, D_k u)_0 = \sum_{j\in\mathbb{N}} (D_j u, D_j u)_0 = \sum_{j\in\mathbb{N}} \|D_j u\|_0^2.$$

$\square$

Furthermore we can prove corresponding approximation and inverse inequality in terms of variants of Jackson's and Bernstein's inequality for the orthogonal projectors D_k.

Lemma 7 (Approximation properties)**.** *We have*

$$\|D_k u\|_0 \leq c h_k^s \|u\|_s, \quad \forall u \in H^s(\Omega)$$

for $0 \leq s \leq 2$.

Proof.

$$\begin{aligned}\|D_k u\|_0 = \|(Q_k - Q_{k-1})u\|_0 = \|(Q_k - Q_k Q_{k-1})u\|_0 \leq \|Q_k\| \|(I - Q_{k-1})u\|_0 \\ \leq c_s h_{k-1}^s \|u\|_s \leq c 2^s h_k^s \|u\|_s\end{aligned}$$

□

Indeed, choosing $u = D_k v \in H^s(\Omega)$ for $0 \leq s < \frac{3}{2}$ we obtain a weaker result.

$$\|D_k v\|_0 \leq c h_k^s \|D_k v\|_s, \quad \forall v \in H^s(\Omega) \tag{5}$$

for $0 \leq s < \frac{3}{2}$. Note, that the structural similarity of (5) to the following inverse inequality allows us to combine these two results in Lemma 9.

Lemma 8 (Inverse inequalities)**.** *We have*

$$\|D_k u\|_s \leq c h_k^{-s} \|D_k u\|_0, \quad \forall u \in L^2(\Omega)$$

for $0 \leq s < \frac{3}{2}$.

Proof. Apply Bernstein's inequality for $D_k u \in M_k$. □

Next we want to combine the approximation and inverse inequalities. Indeed this means to extend the inverse inqualities stated in Lemma 8 to the case $|s| < \frac{3}{2}$. Therefore we have to introduce Sobolev-Slobodecki spaces for negative indices.

Definition 2. *For $s > 0$ the space H^{-s} is defined by the adjoint space of H^s, i.e. $H^{-s} := (H^s)^*$. The associated norm is given by $\|u\|_{-s} = \sup_{\varphi \in H^s} \frac{\langle u, \varphi \rangle}{\|\varphi\|_s}$.*

The next lemma is the combination of the inverse and approximation properties of D_k.

Lemma 9 (Extended inverse inequalities)**.** *We have*

$$\|D_k u\|_s \leq c h_k^{-s} \|D_k u\|_0, \quad \forall u \in L_2(\Omega)$$

for $|s| < \frac{3}{2}$.

Proof. It remains to prove the inequality for negative s. Therefore let $t > 0$.

$$\begin{aligned}\|D_k u\|_{-t} = \sup_{\varphi \in H^t} \frac{\langle D_k u, \varphi \rangle}{\|\varphi\|_t} = \sup_{\varphi \in H^t} \frac{(D_k u, \varphi)_0}{\|\varphi\|_t} = \sup_{\varphi \in H^t} \frac{(D_k u, D_k \varphi)_0}{\|\varphi\|_t} \\ \leq \sup_{\varphi \in H^t} \frac{\|D_k \varphi\|_0}{\|\varphi\|_t} \|D_k u\|_0 \leq c h_k^t \|D_k u\|_0\end{aligned}$$

□

Lemma 10. *We have*

$$(u, v)_s \leq \|u\|_{s+\varepsilon} \|v\|_{s-\varepsilon}, \quad \forall u, v \in H^{s+\varepsilon}(\Omega)$$

for $|s + \varepsilon| < \frac{3}{2}$ *and* $\varepsilon > 0$.

Proof. For this proof we have to recall the representation of scalar products and norms in terms of the spectral decomposition of some generating operator. (For details see e.g. Seminar 05, or the corresponding literature [3]).

Let Λ be the symmetric and positive definite operator defined by the equality

$$\|u\|_2 = \|\Lambda u\|_0.$$

Since Λ is symmetric and positive definite, we have an eigensystem $(\lambda_i, \varphi_i)_{i\in\mathbb{N}}$ of Λ, where $\lambda_i \in \mathbb{R}^+$ and the $\{\varphi_i\}_{i\in\mathbb{N}}$ is orthogonal and complete in $L_2(\Omega)$. Hence

$$u = \sum_{i\in\mathbb{N}} (u, \varphi_i)_0 \varphi_i \quad \Lambda u = \sum_{i\in\mathbb{N}} \lambda_i (u, \varphi_i)_0 \varphi_i$$

Additionally we also have a spectral representation for all $0 \le s \le 2$

$$\begin{aligned}(u, v)_s &= \sum_{k\in\mathbb{N}} \lambda_i^s (u, \varphi_i)_0 (v, \varphi_i)_0 \\ \|u\|_s^2 &= \sum_{k\in\mathbb{N}} \lambda_i^s |(u, \varphi_i)_0|^2.\end{aligned}$$

This technique can also be extended to the case $-2 \le s \le 0$, leading to

$$(u, v)_s = \sum_{k\in\mathbb{N}} \lambda_i^s (u, \varphi_i)_0 (v, \varphi_i)_0, \quad \forall |s| \le 2.$$

Now using this spectral representation, the proof can be done in a very easy manner.

$$\begin{aligned}(u, v)_s &= \sum_{i\in\mathbb{N}} \lambda_i^s (u, \varphi_i)_0 (v, \varphi_i)_0 = \sum_{i\in\mathbb{N}} \lambda_i^{\frac{s+\varepsilon}{2}} (u, \varphi_i)_0 \lambda_i^{\frac{s-\varepsilon}{2}} (v, \varphi_i)_0 \\ &\le \sqrt{\sum_{i\in\mathbb{N}} \lambda_i^{s+\varepsilon} |(u, \varphi_i)_0|^2} \sqrt{\sum_{i\in\mathbb{N}} \lambda_i^{s-\varepsilon} |(v, \varphi_i)_0|^2} = \|u\|_{s+\varepsilon} \|v\|_{s-\varepsilon}\end{aligned}$$

□

Note, that Lemma 10 will also be used in a very specific regime. Choosing $s = 0$ and $\varepsilon = t$, we have $(u, v)_0 \le \|u\|_t \|v\|_{-t}$ for $|t| < \frac{3}{2}$. This result also directly follows by the definition of the dual norm $\|v\|_{-t} = \sup_{u\in H^t} \frac{\langle u,v\rangle}{\|u\|_t} = \sup_{u\in H^t} \frac{(u,v)_0}{\|u\|_t}$.

4.1. **The infinite case.** By considering a weighted linear combination of the orthogonal L_2 projection operators D_k, we define the multilevel operatof B^s

$$B^s := \sum_{k\in\mathbb{N}} h_k^{-2s} D_k$$

which induces an equivalent norm in the Sobolev space $H^s(\Omega)$.

Theorem 1 (Equivalent norm in H^s)**.** *We have*

$$c_1^B \|u\|_s^2 \le (B^s u, u)_0 \le c_2^B \|u\|_s^2, \quad \forall u \in H^s(\Omega)$$

for $|s| < \frac{3}{2}$.

Proof. Representation: Using the fact that D_k is a projection, we easily obtain

$$(B^s u, u)_0 = \sum_{k\in\mathbb{N}} h_k^{-2s} \|D_k u\|_0^2.$$

Upper bound: For proving the upper bound, we are using the projection and orthogonality properties of D_k, Lemma 10 for the special case ('$s = 0$' and '$\varepsilon = s$')

and the extended inverse inequality for negative values.

$$
\begin{aligned}
\sum_{k\in\mathbb{N}} h_k^{-2s}\|D_k u\|_0^2 &= \sum_{k\in\mathbb{N}} h_k^{-2s}(D_k u, D_k u)_0 = \sum_{k\in\mathbb{N}} h_k^{-2s}(u, D_k u)_0 \\
&= (u, \sum_{k\in\mathbb{N}} h_k^{-2s} D_k u)_0 \le \|u\|_s \|\sum_{k\in\mathbb{N}} h_k^{-2s} D_k u\|_{-s} \\
&= \|u\|_s \left(\|\sum_{k\in\mathbb{N}} h_k^{-2s} D_k u\|_{-s}^2 \right)^{\frac{1}{2}} = \|u\|_s \left(\sum_{j\in\mathbb{N}} \|D_j \sum_{k\in\mathbb{N}} h_k^{-2s} D_k u\|_{-s}^2 \right)^{\frac{1}{2}} \\
&\le c\|u\|_s \left(\sum_{j\in\mathbb{N}} h_j^{2s} \|D_j \sum_{k\in\mathbb{N}} h_k^{-2s} D_k u\|_0^2 \right)^{\frac{1}{2}} = c\|u\|_s \left(\sum_{j\in\mathbb{N}} h_j^{2s} \|D_j h_j^{-2s} u\|_0^2 \right)^{\frac{1}{2}} \\
&= c\|u\|_s \left(\sum_{j\in\mathbb{N}} h_j^{-2s} \|D_j u\|_0^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

Chanceling the squareroot term we obtain the upper bound.

Lower bound: For proving the lower bound, we use the decompositon of any L_2 function $u = \sum_{i\in\mathbb{N}} D_k u$. Note, that this decomposition is orthogonal in $L_2(\Omega)$, but not in $H^s(\Omega)$. Furthermore we use Lemma 10, the extended inverse inequality and the L_2 orthogonality of D_k.

$$
\begin{aligned}
\|u\|_s^2 &= (u,u)_s = (\sum_{k\in\mathbb{N}} D_k u, \sum_{j\in\mathbb{N}} D_j u)_s \\
&= \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} (D_k u, D_j u)_s \le \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} \|D_k u\|_{s+\varepsilon}\|D_j u\|_{s-\varepsilon} \\
&\le c\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} h_k^{-(s+\varepsilon)}\|D_k u\|_0 h_j^{-(s-\varepsilon)}\|D_j u\|_0 \\
&= c\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} \left(\frac{h_j}{h_k}\right)^{\varepsilon} h_k^{-s}\|D_k u\|_0 h_j^{-s}\|D_j u\|_0 \\
&\le c\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} 2^{-\varepsilon|j-k|} h_k^{-s}\|D_k u\|_0 h_j^{-s}\|D_j u\|_0
\end{aligned}
$$

By denoting $M = (M_{kj})_{k,j\in\mathbb{N}}$ with $M_{kj} = 2^{-\varepsilon|j-k|}$ and $v = (v_k)_{k\in\mathbb{N}}$ with $v_k = h_k^{-s}\|D_k u\|_0$, we proceed

$$
\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} 2^{-\varepsilon|j-k|} h_k^{-s}\|D_k u\|_0 h_j^{-s}\|D_j u\|_0 = (Mv, v)_{l^2} \le \|M\|_{l^2}\|v\|_{l^2}^2
$$

Here $\|\cdot\|_{l^2}$ is the spectral norm given by $\|M\|_{l^2} = \sqrt{\lambda_{\max}(M^T M)}$, which indeed is the appropriate matrix norm of the Euklidean vector norm, i.e. $\|Ax\|_{l^2} \le \|A\|_{l^2}\|x\|_{l^2}$. Now from Schur's Lemma [5, Lemma 13.17]

$$
\|M\|_{l^2} \le \sup_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} |M_{kj}|
$$

we obtain the final estimate. Using property (3) we can conclude that

$$
\sup_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} |M_{kj}| = \sup_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} 2^{-\varepsilon|j-k|}
$$

Evaluating the geometric series ($q = 2^{-\varepsilon} < 1$ for $\varepsilon > 0$) yields

$$\sum_{k\in\mathbb{N}} q^{|k-j|} = \sum_{k=1}^{j-1} q^{j-k} + \sum_{k=j}^{\infty} q^{k-j} \le 2\sum_{k\in\mathbb{N}} q^k = \frac{2}{1-q}.$$

This finishes the proof. □

The next theorem states, that the decomposition of L_2, that realizes the infimum is essentially $u = \sum_{k\in\mathbb{N}} D_k u$.

Theorem 2. *If $s > 0$, then*

$$\sum_{k\in\mathbb{N}} h_k^{-2s}\|D_k u\|_0^2 \approx \inf_{u=\sum_{k\in\mathbb{N}} u_k} \left\{ \sum_{k\in\mathbb{N}} h_k^{-2s}\|u_k\|_0^2 \right\}$$

Proof. Upper bound: Choosing the specific decomposition $u_k = D_k u$.

Lower bound: For proving the lower bound, we first choose any decompostion $u = \sum_{j\in\mathbb{N}} u_j$.

$$\begin{aligned}\sum_{k\in\mathbb{N}} h_k^{-2s}(D_k u, u)_0 &= \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} h_k^{-2s}(D_k u, u_j)_0 \\ &\le \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} h_k^{-2s}\|D_k u\|_0\|u_j\|_0 = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} \left(\frac{h_j}{h_k}\right)^s h_k^{-s}\|D_k u\|_0 h_j^{-s}\|u_j\|_0\end{aligned}$$

Analogous to the proof of Theorem 1, we can rewrite this double sum as a vector-matrix-vector multiplication and apply Schur's lemma. Hence

$$\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} \left(\frac{h_j}{h_k}\right)^s h_k^{-s}\|D_k u\|_0 h_j^{-s}\|u_j\|_0 \le c\left(\sum_{k\in\mathbb{N}} h_k^{-2s}\|D_k u\|_0^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} h_k^{-2s}\|u_k\|_0^2\right)^{\frac{1}{2}}$$

Chanceling the squareroot term we obtain the lower bound, since the decompostion u_k was arbitrary. □

For the inverse operator $(B^s)^{-1}$ again a multilevel representation can be given.

Theorem 3 (Inverse Operator)**.** *The inverse operator $(B^s)^{-1}$ allows the representation*

$$(B^s)^{-1} = \sum_{k\in\mathbb{N}} h_k^{2s} D_k.$$

Proof.

$$(B^s)^{-1}B^s = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} h_k^{2s}h_j^{-2s} D_k D_j = \sum_{k\in\mathbb{N}} D_k = I$$

□

By using Theorem 1, the multilevel operator B^s is bounded and H^s-elliptic. The inverse operator $(B^s)^{-1}$ is bounded and H^{-s} elliptic. In particular, the spectral equivalence (2) is valid. Hence B^s can be used as a preconditioner resulting in mesh-independent convergence rates in any appropriate iterative method.

4.2. **The finite case.** Let $u_J \in M_J$ with $J >> 1$.

We have

$$D_k u_J = Q_k u_J - Q_{k-1} u_J = \begin{cases} u_J - u_J = 0, & \text{for } k > J \\ D_k u_J. & \text{for } k \leq J \end{cases}$$

By considering a weighted linear combination of the orthogonal L_2 projection operators D_k, we define the finite multilevel operatof B^s

$$B_J^s := \sum_{k=0}^{J} h_k^{-2s} D_k$$

which induces an equivalent norm in the Sobolev space $H^s(\Omega)$.

Corollary 2 (Equivalent norm in M_J)**.** *We have*

$$\|u_j\|_s^2 \approx \sum_{k=1}^{J} h_k^{-2s} \|D_k u_J\|_0^2, \quad \forall u_J \in M_J$$

for $|s| \leq \frac{3}{2}$.

Corollary 3. *If* $s > 0$*, then*

$$\sum_{k=1}^{J} h_k^{-2s} \|D_k u\|_0^2 \approx \inf_{u=\sum_{k=1}^{J} u_k} \left\{ \sum_{k=1}^{J} h_k^{-2s} \|u_k\|_0^2 \right\}$$

Remark 2 (Realization)**.** *For the* $H^1(\Omega)$ *case see e.g.* [5, p. 315-319]

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