Multilevel Representation of the H^s-norm and Preconditioning II

SE: Interpolation Spaces and Applications in Numerical Analysis

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Overview

1. Multilevel setting

2. BPX (Bramble Pasciak Xu) preconditioner

Notations

Hilbert spaces
H^s Sobolev-Slobodecki space of order s

Scalar products
$(\cdot,\cdot)_s = (\cdot,\cdot)_{H^s(\Omega)}$
$(\cdot,\cdot)_0 = (\cdot,\cdot)_{L_2(\Omega)}$

Norms
$\|\cdot\|_s = \|\cdot\|_{H^s(\Omega)}$
$\|\cdot\|_0 = \|\cdot\|_{L_2(\Omega)}$

Duality product
$\langle\cdot,\cdot\rangle$

Geometry and Mesh

Bounded domain $\Omega \subset \mathbb{R}^d$ (here $d = 2$)
Family of meshes $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ with meshsize h_k

- Triangulation with triangles
- global quasi uniform (i.e $\frac{\max_k h_k}{\min_k h_k} \leq c$)
- $c_1 2^{-k} \leq h_k \leq c_2 2^{-k}$ (e.g. dyadic decomposition)

Familiy of nested finite element spaces $\{M_k\}_{k\in\mathbb{N}}$

$$M_1 \subset M_2 \subset M_3 \subset \ldots \subset M_k \subset \ldots$$

where each space M_k corresponds to $S^1_{h_k}$ (set of continuous, piecewise linear functions).
Hence

$$M_k \subset H^{3/2-\varepsilon}(\Omega)$$

for any $\varepsilon > 0$.

L_2-projection

Family of L_2-projection $\{Q_k\}_{k\in\mathbb{N}}$
$Q_k : L_2(\Omega) \rightarrow M_k$ and Q_k defined by

$$(Q_k u, \varphi_k)_0 = (u, \varphi_k)_0, \quad \forall \varphi_k \in M_k,\ \forall u \in L_2(\Omega)$$

Recall

- $Q_k u \in M_k$
- $Q_k Q_j = Q_j Q_k = Q_{\min\{j,\,k\}}$
- $\|Q_k\| = 1$

Lemma (Jackson's (approximation) inequalities)

$$\|(I - Q_k)u\|_0 \leq c h_k^s \|u\|_s, \quad \forall u \in H^s(\Omega)$$

for $0 \leq s \leq 2$.

Note: $\|(I - Q_k)u\|_0 \overset{k\to\infty}{\longrightarrow} 0$ even for $u \in L_2$.

Lemma (Bernstein's (inverse) inequalities)

$$\|u_k\|_s \leq c h_k^{-s} \|u_k\|_0, \quad \forall u_k \in M_k$$

for $0 \leq s < \frac{3}{2}$.

An orthogonal decomposition of L_2

Definition

$$D_1 := Q_1$$
$$D_k := Q_k - Q_{k-1}, \quad k = 2, 3, \ldots$$

We have

1. $D_k u \in M_k$
2. $D_k D_j = 0$ for $k \neq j$.
3. $D_k D_k = D_k$.
4. D_k is self-adjoint

D_k is an orthogonal projector wrt $(\cdot, \cdot)_0$.

Lemma (Orthogonal decomposition of $L_2(\Omega)$)

- $L_2(\Omega) = \sum_{k\in\mathbb{N}} \mathcal{O}_k$, *with* $\mathcal{O}_k = \{\varphi : \varphi = D_k u, u \in L_2(\Omega)\}$
- $u = \sum_{k\in\mathbb{N}} D_k u$ *in* L_2*-sense.*
- $\|u\|_0^2 = \sum_{k\in\mathbb{N}} \|D_k u\|_0^2, \quad \forall u \in L_2(\Omega)$

A version of Jackson's and Bernstein's inequality

Lemma (Approximation properties)

$$\|D_k u\|_0 \leq c h_k^s \|u\|_s, \quad \forall u \in H^s(\Omega)$$

for $0 \leq s \leq 2$*.*

Proof.

$$\begin{aligned} \|D_k u\|_0 &= \|(Q_k - Q_{k-1})u\|_0 = \|(Q_k - Q_k Q_{k-1})u\|_0 \leq \|Q_k\| \|(I - Q_{k-1})u\|_0 \\ &\leq c_s h_{k-1}^s \|u\|_s \leq 2^s h_k^s \|u\|_s \end{aligned}$$

□

Indeed, choosing $u = D_k v \in H^s(\Omega)$ for $0 \leq s < \frac{3}{2}$

$$\|D_k v\|_0 \leq c h_k^s \|D_k v\|_s, \quad \forall v \in H^s(\Omega)$$

for $0 \leq s < \frac{3}{2}$.

Inverse inequalities

Lemma (Inverse inequalities)

$$\|D_k u\|_s \leq c h_k^{-s} \|D_k u\|_0, \quad \forall u \in L^2(\Omega)$$

for $0 \leq s < \frac{3}{2}$.

Proof.

Apply Bernstein's inequality for $D_k u \in M_k$. □

Extension of inverse inqualities

Adjoint space for H^s:

- $H^{-s} := (H^s)^*$
- norm $\|u\|_{-s} = \sup_{\varphi \in H^s} \frac{\langle u, \varphi \rangle}{\|\varphi\|_s}$

Combining Jackson's and Bernstein's inequality.

Lemma (Extended inverse inequalities)

$$\|D_k u\|_s \leq c h_k^{-s} \|D_k u\|_0, \quad \forall u \in L_2(\Omega)$$

for $|s| < \frac{3}{2}$.

Proof.

Let $t > 0$.

$$\begin{aligned} \|D_k u\|_{-t} &= \sup_{\varphi \in H^t} \frac{\langle D_k u, \varphi \rangle}{\|\varphi\|_t} = \sup_{\varphi \in H^t} \frac{(D_k u, \varphi)_0}{\|\varphi\|_t} = \sup_{\varphi \in H^t} \frac{(D_k u, D_k \varphi)_0}{\|\varphi\|_t} \\ &\leq \sup_{\varphi \in H^t} \frac{\|D_k \varphi\|_0}{\|\varphi\|_t} \|D_k u\|_0 \leq c h_k^t \|D_k u\|_0 \end{aligned}$$

$\square$

Recall: Spectral representation (see Meth.- SE 05)

There exists a symmetric positive definite operator Λ, such that

$$\|u\|_2 = \|\Lambda u\|_0.$$

Eigensystem $(\lambda_i, \varphi_i)_{i\in\mathbb{N}}$ of Λ,

$$\Lambda\varphi_i = \lambda_i\varphi_i.$$

where $\lambda_i \in \mathbb{R}^+$ and $\{\varphi_i\}$ orthogonal and complete in L_2, hence

$$u = \sum_{i\in\mathbb{N}}(u,\varphi_i)_0\varphi_i \quad \Lambda u = \sum_{i\in\mathbb{N}}\lambda_i(u,\varphi_i)_0\varphi_i$$

Hence, we have the representation for $0 \leq s \leq 2$

$$(u,v)_s = \sum_{k\in\mathbb{N}}\lambda_i^s(u,\varphi_i)_0(v,\varphi_i)_0$$

$$\|u\|_s^2 = \sum_{k\in\mathbb{N}}\lambda_i^s|(u,\varphi_i)_0|^2$$

This characterization can be extended to the case $|s| \leq 2$.

A nice inequality

Lemma

$$(u, v)_s \leq \|u\|_{s+\varepsilon}\|v\|_{s-\varepsilon}, \quad \forall u, v \in H^{s+\varepsilon}(\Omega)$$

for $|s+\varepsilon| < \frac{3}{2}$ *and* $\varepsilon > 0$.

Proof.

Spectral representation

$$(u, v)_s = \sum_{i\in\mathbb{N}} \lambda_i^s (u, \varphi_i)_0 (v, \varphi_i)_0 = \sum_{i\in\mathbb{N}} \lambda_i^{\frac{s+\varepsilon}{2}} (u, \varphi_i)_0 \lambda_i^{\frac{s-\varepsilon}{2}} (v, \varphi_i)_0$$

$$\leq \sqrt{\sum_{i\in\mathbb{N}} \lambda_i^{s+\varepsilon} |(u, \varphi_i)_0|^2} \sqrt{\sum_{i\in\mathbb{N}} \lambda_i^{s-\varepsilon} |(v, \varphi_i)_0|^2}$$

$$= \|u\|_{s+\varepsilon}\|v\|_{s-\varepsilon}$$

$\square$

Example: Choosing $s = 0$ and $\varepsilon = t$, we have for $|t| < \frac{3}{2}$

$$(u, v)_0 \leq \|u\|_t \|v\|_{-t}$$

Multilevel representation of the H^s norm

Multilevel operator B^s:

$$B^s := \sum_{k \in \mathbb{N}} h_k^{-2s} D_k$$

Theorem (Equivalent norm in H^s)

For $u \in H^s(\Omega)$

$$c_1^B \|u\|_s^2 \leq (B^s u, u)_0 \leq c_2^B \|u\|_s^2$$

for $|s| < \frac{3}{2}$.

Proof: main steps

1. Representation

$$(B^s u, u)_0 = \sum_{k\in\mathbb{N}} h_k^{-2s} \|D_k u\|_0^2$$

Tools: Projection $D_k^2 = D_k$

2. Spectral inequality

$$c_1 \|u\|_s^2 \leq \sum_{k\in\mathbb{N}} h_k^{-2s} \|D_k u\|_0^2 \leq c_2 \|u\|_s^2$$

Tools:

- Nice inequality
- Extended inverse inequality (includes approximation property)
- Schur Lemma

Theorem

If $s > 0$, then

$$\sum_{k\in\mathbb{N}} h_k^{-2s}\|D_k u\|_0^2 \approx \inf_{u=\sum_{k\in\mathbb{N}} u_k} \left\{ \sum_{k\in\mathbb{N}} h_k^{-2s}\|u_k\|_0^2 \right\}$$

The decomposition, that realizes the infimum is essentially $u = \sum_{k\in\mathbb{N}} D_k u$

H^s-bounded and H^s-elliptic operator A

$$c_1\|u\|_s^2 \leq (Au,u)_0 \leq c_2\|u\|_s^2$$

Spectral inequality:

$$c_1(B^s u,u)_0 \leq (Au,u)_0 \leq c_2(B^s u,u)_0$$

for $|s| < \frac{3}{2}$.

Condition number:

$$\kappa((B^s)^{-1}A) \leq \frac{c_2}{c_1}$$

doesn't depend on the mesh parameter h.

Inverse Operator

Theorem (Inverse Operator)

$$(B^s)^{-1} = \sum_{k\in\mathbb{N}} h_k^{2s} D_k$$

Proof.

$$(B^s)^{-1} B^s = \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} h_k^{2s} h_j^{-2s} D_k D_j = \sum_{k\in\mathbb{N}} D_k = I$$

□

Finite element space

Let $u_J \in M_J$ with $J >> 1$.
We have

$$D_k u_J = Q_k u_J - Q_{k-1} u_J = \begin{cases} u_J - u_J = 0, & \text{for } k > J \\ D_k u_J. & \text{for } k \leq J \end{cases}$$

Theorem

$$c_1 \|u_J\|_s^2 \leq \sum_{k=1}^{J} h_k^{-2s} \|D_k u_J\|_0^2 \leq c_2 \|u_J\|_s^2, \quad \forall u_J \in M_J$$

Realization

- Exercise
- see Steinbach for the H^1 case