Alternative approaches

SE: Interpolation Spaces and Applications in Numerical Analysis

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Overview

1. Interpolation in Hilbert spaces
   - Interpolating due to Lions
   - Examples

2. Equivalence to the K-method

- Hilbert spaces X and Y, scalar products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$
- $X \subset Y$, X dense in Y with continuous injection.

Basic idea

The space X can be defined as the domain of an operator Λ, denoted by $D(\Lambda)$, where Λ is self-adjoint and positive definite in Y.

Roadmap for construction

- $D(S)$ is the set of all $u \in Y$, such that the linear form
$$v \longrightarrow (u, v)_X$$
is continuous in the topology induced by Y.
- Hence S is defined and
$$(u, v)_X = (Su, v)_Y$$
- S is self adjoint and positive.
- We can define $\Lambda = S^{\frac{1}{2}}$ and we have
$$(u, v)_X = (\Lambda u, \Lambda v)_Y$$

Definition

The interpolation space is defined by

$$[X, Y]_\theta := D(\Lambda^{1-\theta}), \quad 0 \le \theta \le 1$$

the norm in $[X, Y]_\theta$ is given by the graph norm of $\Lambda^{1-\theta}$,

$$\left(\|u\|_Y^2 + \|\Lambda^{1-\theta} u\|_Y^2\right)^{1/2}$$

Remark

- $[X, Y]_0 = X$ (note: $\|\Lambda u\|_Y = \|u\|_X$)
- $[X, Y]_1 = Y$ (note: $\Lambda^0 = \mathrm{Id}$)
- Λ not unique, but:
  If for two positive and self adjoint operators Λ_1 and Λ_2 in Y with domain X, i.e. $D(\Lambda_1) = D(\Lambda_2) = X$, we have $D(\Lambda_1^{1-\theta}) = D(\Lambda_2^{1-\theta})$ with equivalent norms.

Construction of some S

Injection $\mathcal{I} : X \to Y$ (injective), with

- $\mathcal{I}(X)$ dense in Y, i.e. $\overline{\mathcal{I}(X)} = Y$.
- continuous injection, i.e $\|\mathcal{I}u\|_Y \leq c\|u\|_X$.

Remark

- '$\mathcal{I}u = u$' (note that $X \subset Y$)
- $\mathcal{I}^{-1} : \mathcal{I}(X) \to X$
- $\mathcal{I}^* : Y \to X$ defined by

$$(\mathcal{I}^* w, v)_X = (w, \mathcal{I}v)_Y, \quad w \in Y, v \in X.$$

- $\mathcal{I}\mathcal{I}^* : D(\mathcal{I}\mathcal{I}^*) \to Y$ continuous, positive and self-adjoint.
- $(\mathcal{I}\mathcal{I}^*)^{-1} : R(\mathcal{I}\mathcal{I}^*) \to Y$
- $u \in R(\mathcal{I}\mathcal{I}^*) \Rightarrow u \in R(\mathcal{I}) = D(\mathcal{I}^{-1})$

Definition

We define the operator S by

$$S := (\mathcal{I}\mathcal{I}^*)^{-1}$$

Recall

On the set $D(S)$ the linear form $v \to (u, v)_X$ is continuous in the topology induced by Y.

Lemma

1. *For* $S = (\mathcal{I}\mathcal{I}^*)^{-1}$, *we have* $R(\mathcal{I}\mathcal{I}^*) \subset D(S)$ *and*

$$\forall u \in R(\mathcal{I}\mathcal{I}^*)\ \forall v \in X : \quad (\mathcal{I}^{-1}u, v)_X \leq c\|\mathcal{I}v\|_Y$$

2. $u \in Y : \forall v \in X\ (\mathcal{I}^{-1}u, v)_X \leq c\|\mathcal{I}v\|_Y \Rightarrow u \in R(\mathcal{I}\mathcal{I}^*)$.

Proof.

1. Since $u \in R(\mathcal{I}\mathcal{I}^*)$, we have $\exists w \in Y : u = \mathcal{I}\mathcal{I}^* w$.
   We have the following estimate

$$(\mathcal{I}^{-1}u, v)_X = (\mathcal{I}^* w, v)_X = (w, \mathcal{I}v)_Y \leq \underbrace{\|w\|_Y}_{c} \|\mathcal{I}v\|_Y.$$

2. Choosing $v = \mathcal{I}^{-1}u \in X$ yields

$$((\mathcal{I}\mathcal{I}^*)^{-1}u, u)_Y = (\mathcal{I}^{-1}u, \mathcal{I}^{-1}u)_X \leq c\|u\|_Y$$

   hence $u \in R(\mathcal{I}\mathcal{I}^*)$.

Recall

The following identity holds

$$(u, v)_X = (Su, v)_Y$$

where S is defined on $D(S)$.

Lemma

For $S = (\mathcal{I}\mathcal{I}^*)^{-1}$ we have the identity

$$\forall u \in R(\mathcal{I}\mathcal{I}^*)\ \forall v \in X: \quad (\mathcal{I}^{-1}u, v)_X = (Su, \mathcal{I}v)_Y$$

Proof.

Since $u \in R(\mathcal{I}\mathcal{I}^*)$, we have $\exists w \in Y : u = \mathcal{I}\mathcal{I}^* w$.
For $v \in X$ we have

$$((\mathcal{I}\mathcal{I}^*)^{-1}u, \mathcal{I}v)_Y = ((\mathcal{I}\mathcal{I}^*)^{-1}(\mathcal{I}\mathcal{I}^*)w, \mathcal{I}v)_Y = (w, \mathcal{I}v)_Y = (\mathcal{I}^* w, v)_X = (\mathcal{I}^{-1}u, v)_X.$$

□

Properties of $S = (\mathcal{I}\mathcal{I}^*)^{-1}$

- $D(S)$ is dense in Y.
- self-adjoint on $R(\mathcal{I}\mathcal{I}^*)$

$$(Su, v)_Y = (\mathcal{I}^{-1}u, \mathcal{I}^{-1}v)_X = (\mathcal{I}^{-1}v, \mathcal{I}^{-1}u)_X = (Sv, u)_Y$$

- positive on $R(\mathcal{I}\mathcal{I}^*)$:

$$(Sv, v)_Y = (\mathcal{I}^{-1}v, \mathcal{I}^{-1}v)_X = \|\mathcal{I}^{-1}v\|_X^2 \geq c\|\mathcal{I}\mathcal{I}^{-1}v\|_Y^2 = c\|v\|_Y^2$$

Based on spectral decomposition of self-adjoint operators.

We can define $\Lambda = S^{\frac{1}{2}} = (\mathcal{I}\mathcal{I}^*)^{-\frac{1}{2}}$.

Intermediate result

$$[X, Y]_\theta := D((\mathcal{I}\mathcal{I}^*)^{-\frac{1-\theta}{2}})$$

Assumption

$\mathcal{I}$ is a compact operator.

Singular value decomposition of compact operators (spectral theorem)

Definition

Let $\mathcal{I} : X \to Y$ be compact. A series $(\sigma_n; u_n, v_n)_{n\in\mathbb{N}}$ is called *singular system*, if

- $\sigma_n > 0$
- $(\sigma_n^2, u_n)_{n\in\mathbb{N}}$ is an eigensystem of $\mathcal{I}\mathcal{I}^*$
- $v_n = \frac{\mathcal{I}^* u_n}{\|\mathcal{I}^* u_n\|}$

Important properties

- $\mathcal{I}^* u_n = \sigma_n v_n$
- $\mathcal{I} v_n = \sigma_n u_n$
- $\{u_n\}_{n\in\mathbb{N}}$ ONB for $\overline{R(\mathcal{I}\mathcal{I}^*)} = \overline{R(\mathcal{I})}$
- $\{v_n\}_{n\in\mathbb{N}}$ ONB for $\overline{R(\mathcal{I}^*\mathcal{I})} = \overline{R(\mathcal{I}^*)}$

$$\mathcal{I}\mathcal{I}^* u = \sum_{n\in\mathbb{N}} \sigma_n^2 \alpha_n u_n \qquad (\mathcal{I}\mathcal{I}^*)^{-\frac{1}{2}} u = \sum_{n\in\mathbb{N}} \frac{\alpha_n}{\sigma_n} u_n$$

$$(\mathcal{I}\mathcal{I}^*)^{-1} u = \sum_{n\in\mathbb{N}} \frac{\alpha_n}{\sigma_n^2} u_n \qquad (\mathcal{I}\mathcal{I}^*)^{-\frac{1-\theta}{2}} u = \sum_{n\in\mathbb{N}} \frac{\alpha_n}{\sigma_n^{1-\theta}} u_n$$

where $\alpha_n = (u, u_n)_Y$

Norm representation

$$\|\Lambda^{1-\theta}u\|_Y^2 = ((\mathcal{I}\mathcal{I}^*)^{-\frac{1-\theta}{2}}u, (\mathcal{I}\mathcal{I}^*)^{-\frac{1-\theta}{2}}u)_Y = (\sum_{n\in\mathbb{N}} \frac{\alpha_n}{\sigma_n^{1-\theta}} u_n, \sum_{n\in\mathbb{N}} \frac{\alpha_n}{\sigma_n^{1-\theta}} u_n)_Y$$

$$= \sum_{n,m\in\mathbb{N}} \frac{\alpha_n}{\sigma_n^{1-\theta}} \frac{\alpha_m}{\sigma_m^{1-\theta}} (u_n, u_m)_Y = \sum_{n\in\mathbb{N}} \left(\frac{\alpha_n}{\sigma_n^{(1-\theta)}}\right)^2$$

Interpolation space

In this context the interpolation space can be specified

$$[X, Y]_\theta := D(\Lambda^{1-\theta}) = D((\mathcal{I}\mathcal{I}^*)^{-\frac{1-\theta}{2}})$$

$$= \left\{ u \in Y : \|\Lambda^{1-\theta}u\|_Y^2 = \sum_{n\in\mathbb{N}} \left(\frac{\alpha_n}{\sigma_n^{(1-\theta)}}\right)^2 < \infty \right\}$$

where $u \in Y$ has the decomposition $u = \sum_{n\in\mathbb{N}} \alpha_n u_n$ with $\alpha_n = (u, u_n)_Y$.

$H_0^1 \rightarrow L_2$

- Hilberspace $X = H_0^1(0,1)$
- Norm: $\|u\|_X^2 = \int_0^1 |\nabla u|^2 \mathrm{dx}$
- ONB: $(v_n) = \left(\frac{\sqrt{2}}{n\pi}\sin(n\pi x)\right)_{n\in\mathbb{N}}$

- Hilberspace $Y = L_2(0,1)$
- Norm: $\|u\|_Y^2 = \int_0^1 |u|^2 \mathrm{dx}$
- ONB: $(u_n) = \left(\sqrt{2}\sin(n\pi x)\right)_{n\in\mathbb{N}}$

The injection $\mathcal{I} : H_0^1(0,1) \rightarrow L^2(0,1)$

- continuous (Friedrich's inequality: $\|u\|_Y \leq c\|u\|_X$)
- compact (Rellich Compactness Theorem)
- dense

Singular system

$$\mathcal{I} v_n = v_n = \frac{1}{\pi n}\sqrt{2}\sin(n\pi x) = \sigma_n u_n$$

with $\sigma_n = \frac{1}{n\pi}$.

Interpolation space

$$[X,Y]_\theta = \left\{ u \in Y : \|\Lambda^{1-\theta} u\|_Y^2 = \sum_{n\in\mathbb{N}} \left((n\pi)^{1-\theta}\alpha_n\right)^2 < \infty \right\}$$

with $\alpha_n = \int_0^1 u\sin(n\pi x)\mathrm{dx}$ for $n = \mathbb{N}_0$.

$H^1 \to L_2$

- Hilberspace $X = H^1(0,1)$
- Norm: $\|u\|_X^2 = \int_0^1 |\nabla u|^2 \mathrm{dx} + \left(\int_0^1 u \,\mathrm{dx}\right)^2$
- ONB: $(v_n) = \left(1, \frac{\sqrt{2}}{n\pi}\cos(n\pi x)\right)_{n\in\mathbb{N}}$

- Hilberspace $Y = L_2(0,1)$
- Norm: $\|u\|_Y^2 = \int_0^1 |u|^2 \mathrm{dx}$
- ONB: $(u_n) = \left(1, \sqrt{2}\cos(n\pi x)\right)_{n\in\mathbb{N}}$

The injection $\mathcal{I} : H^1(0,1) \to L^2(0,1)$

- continuous (Sobolev norm equivalence theorem: $\|u\|_Y \le c\|u\|_X$)
- compact
- dense

Singular system

$$\mathcal{I}v_n = v_n = \frac{1}{\pi n}\sqrt{2}\cos(n\pi x) = \sigma_n u_n \quad \text{and} \quad \mathcal{I}v_0 = 1 = \sigma_0 u_0$$

where $\sigma_n = \frac{1}{n\pi}$ and $\sigma_0 = 1$.

Interpolation space

$$[X, Y]_\theta = \left\{ u \in Y : \|\Lambda^{1-\theta} u\|_Y^2 = \alpha_0^2 + \sum_{n\in\mathbb{N}} \left((n\pi)^{1-\theta}\alpha_n\right)^2 < \infty \right\}$$

with $\alpha_n = \int_0^1 u \cos(n\pi x)\mathrm{dx}$ for $n = \mathbb{N}_0$.

Equivalence to the K-method

$$K(t;u) := \inf_{u=u_0+u_1} \left(\|u_0\|_X^2 + t^2\|u_1\|_Y^2\right)^{1/2}$$

Theorem

Let $X \subset Y$ two Hilberspaces, X dense in Y with continuous injection and $0 < \theta < 1$.

1. *The interpolation spaces can identified.*

$$[X,Y]_\theta = \{u : u \in Y, t^{-(\theta+\frac{1}{2})}K(t;u) \in L^2(0,\infty)\}$$

2. *The norms are equivalent.*

$$\|u\|_Y^2 + \|\Lambda^{1-\theta}u\|_Y^2 \sim \|u\|_Y^2 + \int_0^\infty t^{-(2\theta+1)}K(t;u)^2\,\mathrm{dt}$$

Remark

The same holds for K replaced by $\tilde{K}$

$$\tilde{K}(t;u) := \inf_{u=u_0+u_1} \left(\|u_0\|_X + t\|u_1\|_Y\right)$$

Note: $\left(\|u_0\|_X^2 + t^2\|u_1\|_Y^2\right) \le \left(\|u_0\|_X + t\|u_1\|_Y\right)^2 \le 2\left(\|u_0\|_X^2 + t^2\|u_1\|_Y^2\right)$

Proof (sketch).

We have the representation

$$K(t;u)^2 = \inf_{u=u_0+u_1}\left(\|u_0\|_X^2 + t^2\|u_1\|_Y^2\right) = \inf_{u_0\in D(\Lambda)}\left(\|\Lambda u_0\|_Y^2 + t^2\|u-u_0\|_Y^2\right)$$

The corresponding Euler equation of this minimization problem is

$$(\Lambda^2 + t^2)u_0 = t^2 u \quad \text{in } Y$$

Hence we have

$$K(t;u)^2 = t^2\|u\|_Y^2 - t^2(u,u_0)_Y = t^2(u,u-u_0)_Y = (\Lambda^2 u_0, u)_Y = (\Lambda^2 t^2(\Lambda^2+t^2)^{-1}u,u)_Y$$

Using this representation we can compute on the basis of spectral theory

$$\int_0^\infty t^{-(2\theta+1)}K(t,u)^2\,\mathrm{dt} = \underbrace{\left(\int_0^\infty \frac{s^{1-2\theta}}{1+s^2}\mathrm{ds}\right)}_{c}(\Lambda^{2(1-\theta)}u,u)_Y = c\|\Lambda^{1-\theta}u\|_Y^2.$$

□