Interpolation Spaces

The Reiteration Theorem & The Duality Theorem

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Outline

Repetition

Let X_0 and X_1 be Banach spaces imbedded in a normed vector space.

Definition 1 (The J and K norms)

For each fixed $t > 0$ we define the following functionals on $X_0 \cap X_1$ and $X_0 + X_1$ respectively:

$$\begin{aligned} J(t;u) &:= \max\{||u||_{X_0}, t||u||_{X_1}\} \\ K(t;u) &:= \inf_{u=u_0+u_1}\{||u_0||_{X_0} + t||u_1||_{X_1}\} \end{aligned}$$

These functionals define norms on the corresponding spaces.

Repetition contd.

Theorem 2 (The K Method)

If and only if either $1 \leq q < \infty$ *and* $0 < \Theta < 1$ *or* $q = \infty$ *and* $0 \leq \Theta \leq 1$, *then the space* $(X_0, X_1)_{\Theta,q;K}$ *is a nontrivial Banach space with norm:*

$$||u||_{\Theta,q;K} := \begin{cases} \left(\int_0^\infty \left(t^{-\Theta} K(t;u)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty \\ \operatorname{ess\,sup}_{0<t<\infty} \{t^{-\Theta} K(t;u)\} & \text{if } q = \infty. \end{cases}$$

Furthermore,

$$||u||_{X_0+X_1} \leq \frac{||u||_{\Theta,q;K}}{||t^{-\Theta} \min\{1,t\}||_{L^q_*}} \leq ||u||_{X_0 \cap X_1}$$

so there hold the imbeddings

$$X_0 \cap X_1 \longrightarrow (X_0, X_1)_{\Theta,q;K} \longrightarrow X_0 + X_1$$

and $(X_0, X_1)_{\Theta,q;K}$ *is an intermediate space between* X_0 *and* X_1.

Repetition contd.

There is also a discrete Version of the K Method:

Theorem 3 (The Discrete K Method)

For each integer i let $K_i(u) := K(2^i; u)$. Then $u \in (X_0, X_1)_{\Theta,q;K}$ if and only if the sequence $\left(2^{-i\Theta} K_i(u)\right)_{i=-\infty}^{\infty}$ belongs to the space l^q. Moreover, the l^q-norm of that sequence is equivalent to $||u||_{\Theta,q;K}$.

Repetition contd.

Theorem 4 (The J Method)

If either $1 < q \leq \infty$ *and* $0 < \Theta < 1$ *or* $q = 1$ *and* $0 \leq \Theta \leq 1$*, then the space* $(X_0, X_1)_{\Theta,q;J}$ *is a nontrivial Banach space with norm:*

$$||u||_{\Theta,q;J} := \inf_{f \in S(u)} ||t^{-\Theta} J(t; f(t))||_{L^q_*}$$

where

$$S(u) := \left\{ f \in L^1(0, \infty; dt/t, X_0 + X_1) : u = \int_0^\infty f(t) \frac{dt}{t} \right\}.$$

Furthermore,

$$||u||_{X_0+X_1} \leq ||t^{-\Theta} \min\{1, t\}||_{L^q_*} ||u||_{\Theta,q;J} \leq ||u||_{X_0 \cap X_1}$$

so there hold the imbeddings

$$X_0 \cap X_1 \longrightarrow (X_0, X_1)_{\Theta,q;J} \longrightarrow X_0 + X_1$$

and $(X_0, X_1)_{\Theta,q;J}$ *is an intermediate space between* X_0 *and* X_1*.*

Repetition contd.

There is also a discrete Version of the J Method:

Theorem 5 (The Discrete J Method)

An element $u \in X_0 + X_1$ belongs to $(X_0, X_1)_{\Theta,q;J}$ if and only if $u = \sum_{i=-\infty}^{\infty} u_i$ where the series converges in $X_0 + X_1$ and the sequence $\left(2^{-i\Theta} J(2^i; u_i)\right)_{i=-\infty}^{\infty}$ belongs to the space l^q. In this case

$$\inf \left\{ ||2^{-i\Theta} J(2^i; u_i)||_{l^q} : u = \sum_{i=-\infty}^{\infty} u_i \right\}$$

is a norm on $(X_0, X_1)_{\Theta,q;J}$ equivalent to $||u||_{\Theta,q;J}$.

Repetition contd.

The following theorem guarantees that for $0 < \Theta < 1$ the J and K Methods generate the same intermediate spaces with equivalent norms:

Theorem 6 (The Equivalence Theorem)

If $0 < \Theta < 1$ *and* $1 \leq q \leq \infty$ *then* $(X_0, X_1)_{\Theta,q;J} = (X_0, X_1)_{\Theta,q;K}$ *with equivalence of norms.*

Definition 7 (Classes of Intermediate Spaces)

We define three classes of intermediate spaces X between X_0 and X_1 as follows:

1. X belongs to class $\mathcal{K}(\Theta; X_0, X_1)$ if for all $u \in X$

$$K(t; u) \leq C_1 t^{\Theta} ||u||_X$$

   with a constant C_1.

2. X belongs to class $\mathcal{J}(\Theta; X_0, X_1)$ if for all $u \in X_0 \cap X_1$

$$||u||_X \leq C_2 t^{-\Theta} J(t; u)$$

   with a constant C_2.

3. X belongs to class $\mathcal{H}(\Theta; X_0, X_1)$ if X belongs to both $\mathcal{K}(\Theta; X_0, X_1)$ and $\mathcal{J}(\Theta; X_0, X_1)$.

The following theorem gives the result of constructing intermediate spaces between two intermediate spaces:

Theorem 8 (The Reiteration Theorem)

Let $0 \leq \Theta_0 < \Theta_1 \leq 1$ and let X_{Θ_0} and X_{Θ_1} be intermediate spaces between X_0 and X_1. For $0 \leq \lambda \leq 1$ let $\Theta = (1-\lambda)\Theta_0 + \lambda\Theta_1$.

1. *If $X_{\Theta_i} \in \mathcal{K}(\Theta_i; X_0, X_1)$ for $i = 0, 1$, and if either $0 < \lambda < 1$ and $1 \leq q < \infty$ or $0 \leq \lambda \leq 1$ and $q = \infty$, then*

$$(X_{\Theta_0}, X_{\Theta_1})_{\lambda,q;K} \longrightarrow (X_0, X_1)_{\Theta,q;K}.$$

2. *If $X_{\Theta_i} \in \mathcal{J}(\Theta_i; X_0, X_1)$ for $i = 0, 1$, and if either $0 < \lambda < 1$ and $1 < q \leq \infty$ or $0 \leq \lambda \leq 1$ and $q = 1$, then*

$$(X_0, X_1)_{\Theta,q;J} \longrightarrow (X_{\Theta_0}, X_{\Theta_1})_{\lambda,q;J}.$$

3. *If $X_{\Theta_i} \in \mathcal{H}(\Theta_i; X_0, X_1)$ for $i = 0, 1$, and if $0 < \lambda < 1$ and $1 \leq q \leq \infty$, then*

$$(X_{\Theta_0}, X_{\Theta_1})_{\lambda,q;J} = (X_{\Theta_0}, X_{\Theta_1})_{\lambda,q;K} = (X_0, X_1)_{\Theta,q;K} = (X_0, X_1)_{\Theta,q;J}.$$

proof:
Notation: K,J in the construction of intermediate spaces between X_0 and X_1.
K^+,J^+ in the construction of intermediate spaces between X_{Θ_0} and X_{Θ_1}.

1:
Let $u \in (X_{\Theta_0}, X_{\Theta_1})_{\lambda,q;K} \Longrightarrow u = u_0 + u_1, u_i \in X_{\Theta_i}$. Since $X_{\Theta_i} \in \mathcal{K}(\Theta_i; X_0, X_1)$:

$$\begin{aligned} K(t;u) &\leq K(t;u_0) + K(t;u_1) \\ &\leq C_0 t^{\Theta_0} ||u_0||_{X_{\Theta_0}} + C_1 t^{\Theta_1} ||u_1||_{X_{\Theta_1}} \\ &\leq C_0 t^{\Theta_0} K^+\left(\frac{C_1}{C_0} t^{\Theta_1 - \Theta_0}; u\right). \end{aligned}$$

If $\Theta = (1-\lambda)\Theta_0 + \lambda\Theta_1$, then $\lambda = \frac{\Theta - \Theta_0}{\Theta_1 - \Theta_0}$ and

$$||t^{-\Theta} K(t;u)||_{L^q_*} \leq \frac{C_0^{1-\lambda} C_1^{\lambda}}{(\Theta_1 - \Theta_0)^{1/q}} ||u||_{\lambda,q;K} \quad \text{for } q < \infty$$

and

$$||t^{-\Theta} K(t;u)||_{L^q_*} \leq C_0^{1-\lambda} C_1^{\lambda} ||u||_{\lambda,q;K} \quad \text{for } q = \infty$$

via the transformation $s = (C_1/C_0)t^{\Theta_1 - \Theta_0}$.

2:
Let $u \in (X_0, X_1)_{\Theta,q;J}$. Then $u = \int_0^\infty f(s)\frac{ds}{s}$ for some f taking values in $X_0 \cap X_1$, satisfying $s^{-\Theta}J(s; f(s)) \in L_*^q$. Clearly $f(s) \in X_{\Theta_0} \cap X_{\Theta_1}$. Since $X_{\Theta_i} \in \mathcal{J}(\Theta_i; X_0, X_1)$:

$$\begin{aligned} J^+(s; f(s)) &\leq \max\left\{C_0 t^{-\Theta_0} J(t; f(s)), C_1 t^{-\Theta_1} sJ(t; f(s))\right\} \\ &= C_0 t^{-\Theta_0} \max\left\{1, \frac{C_1}{C_0} t^{-(\Theta_1 - \Theta_0)} s\right\} J(t; f(s)) \end{aligned}$$

.

Now choose t such that $t^{-(\Theta_1 - \Theta_0)} s = \frac{C_0}{C_1}$ and obtain

$$J^+(s; f(s)) \leq C_0 \left(\frac{C_1}{C_0} s\right)^{\frac{-\Theta_0}{\Theta_1 - \Theta_0}} J\left(\left(\frac{C_1}{C_0} s\right)^{\frac{1}{\Theta_1 - \Theta_0}}; f(s)\right).$$

If $\Theta = (1-\lambda)\Theta_0 + \lambda\Theta_1$, then $\lambda = \frac{\Theta - \Theta_0}{\Theta_1 - \Theta_0}$ and

$$||s^{-\lambda} J^+(s; f(s))||_{L_*^q} \leq \frac{C_0^{1-\lambda} C_1^{\lambda}}{(\Theta_1 - \Theta_0)^{(q-1)/q}} ||u||_{\Theta,q;J} \quad \text{for } q < \infty$$

and

$$||s^{-\lambda} J^{+}(s; f(s))||_{L_*^q} \leq \frac{C_0^{1-\lambda} C_1^{\lambda}}{(\Theta_1 - \Theta_0)} ||u||_{\Theta,q;J} \quad \text{for } q = \infty$$

via the transformation $g(t) = f(\frac{C_0}{C_1} t^{\Theta_1 - \Theta_0}) = f(s)$, since

$$\int_0^{\infty} g(t) \frac{dt}{t} = \frac{1}{\Theta_1 - \Theta_0} u.$$

3 follows from 1 and 2 $\square$.

From now on we just write $(X_0, X_1)_{\Theta,q}$ for $(X_0, X_1)_{\Theta,q;K} = (X_0, X_1)_{\Theta,q;J}$.

Here we determine the dual $(X_0, X_1)^*_{\Theta,q}$ of the interpolation space $(X_0, X_1)_{\Theta,q}$ when $1 \leq q < \infty$. For proving the duality theorem we first need a few technical results:

Theorem 9

Suppose that $X_0 \cap X_1$ is dense in X_0 and X_1. Then $(X_0 \cap X_1)^* = X_0^* + X_1^*$ and $(X_0 + X_1)^* = X_0^* \cap X_1^*$. More precisely

$$||u^*||_{X_0^*+X_1^*} = \sup_{u \in X_0 \cap X_1} \frac{|\langle u^*, u\rangle|}{||u||_{X_0 \cap X_1}}$$

and

$$||u^*||_{X_0^* \cap X_1^*} = \sup_{u \in X_0 + X_1} \frac{|\langle u^*, u\rangle|}{||u||_{X_0 + X_1}}$$

where $\langle .,. \rangle$ denotes the duality pairing.

Theorem 10

Let X_0, X_1 be a given couple of Banach spaces. Then $(X_0, X_1)_{\Theta,q} = (X_1, X_0)_{1-\Theta,q}$

proof:

1.

$$\begin{aligned} K(t;u) &= K(t;u,X_0,X_1) = \inf_{u=u_0+u_1} \{||u_0||_{X_0} + t||u_1||_{X_1}\} \\ &= t \inf_{u=u_0+u_1} \left\{||u_1||_{X_1} + \frac{1}{t}||u_0||_{X_0}\right\} = tK(t^{-1};u,X_1,X_0) \end{aligned}$$

2.

$$\begin{aligned} ||t^{-\Theta}\phi(t)||_{L^q_*} &= \left(\int_0^\infty \left(t^{-\Theta}\phi(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(t^{\Theta}\phi(t^{-1})\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} = ||t^{\Theta}\phi(t^{-1})||_{L^q_*} \end{aligned}$$

Therefore:

$$||t^{-\Theta}K(t;u,X_0,X_1)||_{L_*^q} = ||t^{\Theta}K(t^{-1};u,X_0,X_1)||_{L_*^q} = ||t^{-(1-\Theta)}K(t;u,X_1,X_0)||_{L_*^q}$$

$\square$.

As a consequence of Theorem 9 we have:

Corollary 11

Assume that $X_0 \cap X_1$ is dense in X_0 and X_1. For $t > 0$, the dual space of $X_0 \cap X_1$ equipped with the norm $J(t;u,X_0,X_1)$ is $X_0^* + X_1^*$ equipped with the norm $K(t^{-1};u^*,X_0^*,X_1^*)$. More precisely:

$$K(t^{-1};u^*,X_0^*,X_1^*) = \sup_{u \in X_0 \cap X_1} \frac{|\langle u^*,u\rangle|}{J(t;u,X_0,X_1)} \tag{1}$$

and

Corollary 12

Assume that $X_0 \cap X_1$ is dense in X_0 and X_1. For $t > 0$, the dual space of $X_0 + X_1$ equipped with the norm $K(t; u, X_0, X_1)$ is $X_0^* \cap X_1^*$ equipped with the norm $J(t^{-1}; u^*, X_0^*, X_1^*)$. More precisely:

$$J(t^{-1}; u^*, X_0^*, X_1^*) = \sup_{u \in X_0 + X_1} \frac{|\langle u^*, u \rangle|}{K(t; u, X_0, X_1)}. \tag{2}$$

Now we can show the following theorem:

Theorem 13 (The Duality Theorem)

Let $\{X_0, X_1\}$ be a couple of Banach spaces, such that $X_0 \cap X_1$ is dense in X_0 and X_1. Assume that $1 \leq q < \infty$ and $0 < \Theta < 1$. Then

$$(X_0, X_1)^*_{\Theta,q} = (X_0^*, X_1^*)_{\Theta,q^*} \quad \textit{(with equivalent norms)}$$

where $\frac{1}{q} + \frac{1}{q^*} = 1$.

proof:
If we prove

1. $(X_0, X_1)^*_{\Theta,q;J} \longrightarrow (X_1^*, X_0^*)_{1-\Theta,q^*;K}$
2. $(X_0^*, X_1^*)_{\Theta,q^*;J} \longrightarrow (X_0, X_1)^*_{\Theta,q;K}$

we get the result by the Equivalence Theorem and Theorem 10.

1:
Take $u^* \in (X_0, X_1)^*_{\Theta,q;J}$ and apply formula (1). Thus, given $\epsilon > 0$, we can find $u_i \in X_0 \cap X_1$ such that $u_i \neq 0$ and, since $u^* \in (X_0 \cap X_1)^* = X_0^* + X_1^*$,

$$K(2^{-i}; u^*, X_0^*, X_1^*) - \epsilon \min\left\{1, 2^{-i}\right\} \leq \frac{\langle u^*, u_i\rangle}{J(2^i; u, X_0, X_1)}.$$

Choose a sequence (α_i) such that $\left(2^{-i\Theta}\alpha_i\right) \in l^q$ and set

$$u_\alpha := \sum_{i=-\infty}^{\infty} J(2^i; u, X_0, X_1)^{-1}\alpha_i u_i$$

then $u_\alpha \in (X_0, X_1)_{\Theta,q;J}$.
Now

$$\langle u^*, u_\alpha\rangle \geq \sum_{i=-\infty}^{\infty} \alpha_i \left(K(2^{-i}; u^*, X_0^*, X_1^*) - \epsilon \min\left\{1, 2^{-i}\right\}\right)$$

and

$$\langle u^*, u_\alpha \rangle \le ||u^*||_{(X_0,X_1)^*_{\Theta,q;J}} ||2^{-i\Theta}\alpha_i||_{l^q}$$

since

$$||u_\alpha||_{\Theta,q;J} \le ||2^{-i\Theta}\alpha_i||_{l^q}.$$

Since

$$\sum_{i=-\infty}^{\infty} \alpha_i\beta_i \le M||2^{-i\Theta}\alpha_i||_{l^q} \Longleftrightarrow ||2^{i\Theta}\beta_i||_{l^{q^*}} \le M$$

and ϵ is arbitrary, the statement follows.

2:
Let $u^* \in (X_0^*, X_1^*)_{\Theta,q^*;J}$ then $u^* = \sum_{i=-\infty}^{\infty} u_i^*$ with $u_i^* \in X_0^* \cap X_1^*$ and $(2^{-i\Theta}J(2^i; u_i^*, X_0^*, X_1^*)) \in l^{q^*}$. For $u \in (X_0, X_1)_{\Theta,q;K}$ we have:

$$|\langle u^*, u \rangle| \le \sum_{i=-\infty}^{\infty} |\langle u_i^*, u \rangle| \le \sum_{i=-\infty}^{\infty} 2^{-i\Theta} J(2^i; u_i^*, X_0^*, X_1^*) 2^{i\Theta} K(2^{-i}; u_i, X_0, X_1)$$

due to formula (2). Applying Hölder's inequality yields the statement $\square$.