Beweis von Lemma 2.16:

$\|\cdot\|_1 := \|\cdot\|_{H^1(a,b)}$, $\|\cdot\|_0 := \|\cdot\|_{L_2(a,b)}$

$$\|u-\tilde{u}_h\|_1^2 = \|\overbrace{u-\tilde{u}_h}^{z\in V_0}\|_0^2 + |u-\tilde{u}_h|_1^2$$

$z = u-\tilde{u}_h \in V_0$, da $z(a) = u(a)-\tilde{u}_h(a) = 0$

$$|u-\tilde{u}_h|_1^2 := \int_a^b [(u-\tilde{u}_h)']^2\,dx = \|(u-\tilde{u}_h)'\|_0^2$$

H^1-Halbnorm

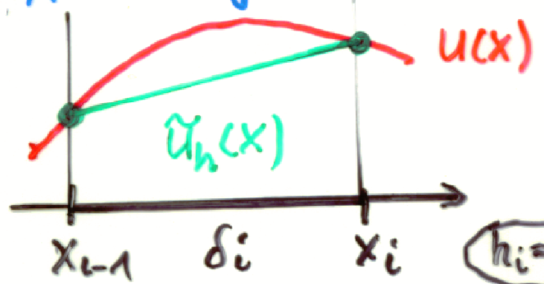
Friedrichsungl. (vgl. Abs. 2.1)

$$\|u-\tilde{u}_h\|_0^2 \le C_F^2\,|u-\tilde{u}_h|_1^2$$

$$\|u-\tilde{u}_h\|_1^2 \le (C_F^2+1)\,|u-\tilde{u}_h|_1^2$$

$$|u-\tilde{u}_h|_1^2 = |z|_1^2 = \int_a^b (z'(x))^2\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (z'(x))^2\,dx$$

Offenbar gilt für den Interpolationsfehler $z = u-\tilde{u}_h$:



$$z(x_j) = u(x_j) - \tilde{u}_h(x_j) = 0 \quad \forall j = \overline{0,n}$$

$h_i = x_i - x_{i-1}$

Folglich gilt:

$$\int_{x_{i-1}}^{x_i} |z'(x)|^2\,dx = \int_{x_{i-1}}^{x_i} \Big| z'(x) - \overbrace{\frac{1}{h_i}\underbrace{\int_{x_{i-1}}^{x_i} z'(\xi)\,d\xi}_{= z(x_i)-z(x_{i-1}) = 0}}^{=0} \Big|^2\,dx$$

$$z'(x) = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} z'(x)\,d\xi$$

$$= \int_{x_{i-1}}^{x_i} \Big[\frac{1}{h_i}\int_{x_{i-1}}^{x_i} \big(\underbrace{z'(x)-z'(\xi)}_{=\int_\xi^x z''(\eta)\,d\eta}\big)\,d\xi\Big]^2\,dx$$

$$= \int_{x_{i-1}}^{x_i} \Big[\frac{1}{h_i}\int_{x_{i-1}}^{x_i} 1\cdot\int_\xi^x z''(\eta)\,d\eta\,d\xi\Big]^2\,dx = (*)$$