Preconditioned Conjugate Gradients (PCG)

Assume A, C symmetric positive definite (SPD) $\implies$ $C^{-1}A$ self-adjoint w.r.t. $(\cdot,\cdot)_C$. Apply CG to the system

$$C^{-1}A\,x \;=\; C^{-1}b$$

Algorithm 1: PCG (theoretical)

x_0 given
$s_0 = C^{-1}b - C^{-1}A\,x_0$
$k = 0$
repeat
  if $k = 0$ **then**
    $p_k = s_k$
  else
    $\beta_{k-1} = \frac{(s_k, s_k)_C}{(s_{k-1}, s_{k-1})_C}$
    $p_k = s_k + \beta_{k-1}\,p_{k-1}$
  end
  $\alpha_k = \frac{(s_k, s_k)_C}{(C^{-1}A\,p_k, p_k)_C}$
  $x_{k+1} = x_k + \alpha_k\,p_k$
  $s_{k+1} = s_k - \alpha_k\,C^{-1}A\,p_k \qquad = C^{-1}b - C^{-1}A\,x_{k+1}$
  $k = k + 1$
until *stopping criterion fulfilled*

Algorithm 2: PCG (practical)

x_0 given
$r_0 = b - A\,x_0$
$k = 0$
repeat
  $s_k = C^{-1}r_k$
  if $k = 0$ **then**
    $p_k = s_k$
  else
    $\beta_{k-1} = \frac{(r_k, s_k)_{\ell^2}}{(r_{k-1}, s_{k-1})_{\ell^2}}$
    $p_k = s_k + \beta_{k-1}\,p_{k-1}$
  end
  $\alpha_k = \frac{(r_k, s_k)_{\ell^2}}{(A\,p_k, p_k)_{\ell^2}}$
  $x_{k+1} = x_k + \alpha_k\,p_k$
  $r_{k+1} = r_k - \alpha_k\,A\,p_k \qquad = b - A\,x_{k+1}$
  $k = k + 1$
until *stopping criterion fulfilled*