

Improved Euler / Explicit midpoint rule

$$\begin{aligned} g_1 &= u_j \\ g_2 &= u_j + \frac{\tau_j}{2} f(t_j, g_1) \\ u_{j+1} &= u_j + \tau_j f(t_j + \frac{\tau_j}{2}, g_2) \end{aligned}$$

In short:

$$\begin{aligned} u_{j+1} &= u_j + \tau_j \phi(t_j, u_j, \tau_j) \quad \text{where} \quad \phi(t, u, \tau) := f\left(t + \frac{\tau}{2}, u + \frac{\tau}{2} f(t, u)\right) \\ \frac{1}{\tau_j} (u_{j+1} - u_j) &= \phi(t_j, u_j, \tau_j) \end{aligned}$$

Consistency analysis

$$\text{Local error: } d_\tau(t + \tau) := u(t + \tau) - [u(t) + \tau \phi(t, u(t), \tau)]$$

$$\text{Consistency error: } \psi_\tau(t_k) = \psi_k \text{ with}$$

$$\begin{aligned} \psi_{j+1} &= \frac{1}{\tau_j} (u(t_{j+1}) - u(t_j)) - \phi(t_j, u(t_j), \tau_j) \\ \psi_0 &= u(0) - u_0 = 0 \end{aligned}$$

For simplicity: dimension $n = 1$, assume that f is sufficiently smooth

$$\begin{aligned} d_\tau(t + \tau) &= u(t + \tau) - [u(t) + \tau \phi(t, u(t), \tau)] \\ &= u(t + \tau) - u(t) - \tau f\left(t + \frac{\tau}{2}, u(t) + \frac{\tau}{2} f(t, u(t))\right) \\ &= u(t) + \tau u'(t) + \frac{\tau^2}{2} u''(t) + \frac{\tau^3}{6} u'''(t) + \mathcal{O}(\tau^4) - u(t) \\ &\quad - \tau \left[\left(f + \frac{\tau}{2} f_t + \frac{\tau}{2} f_u f + \frac{\tau^2}{4} f_{tt} + \frac{\tau^2}{2} f_{tu} f + \frac{\tau^2}{4} f_{uu} f^2 \right)_{(t, u(t))} + \mathcal{O}(\tau^3) \right] \\ &= \tau \left[\underbrace{u'(t) - f(t, u(t))}_{=0} \right] + \frac{\tau^2}{2} \left[\underbrace{u''(t) - (f_t + f_u f)_{(t, u(t))}}_{= (*) = 0} \right] + \\ &\quad + \tau^3 \left[\frac{1}{6} u'''(t) - \frac{1}{4} (f_{tt} + 2f_{tu} f + f_{uu} f^2)_{(t, u(t))} \right] + \mathcal{O}(\tau^4) \end{aligned}$$

where

$$0 = \frac{d}{dt} [u'(t) - f(t, u(t))] = u''(t) - (f_t(t, u(t)) + f_u(t, u(t)) \underbrace{u'(t)}_{=f(t, u(t))}) = (*)$$

$$\text{Recall that } \psi_\tau(u)(t_{j+1}) = \frac{1}{\tau_j} d_\tau(t_{j+1}) \implies$$

$$\psi_\tau(u)(t_{j+1}) = \mathcal{O}(\tau^2).$$

A detailed analysis shows that

$$\exists K > 0 : \|\psi_\tau(t)\|_{Y_\tau} \leq K \tau^2.$$

Summary: The improved Euler method has **consistency order 2**.