1. Implicit Euler method

$$u_{j+1} \;=\; u_j + \tau_j\,\phi(t_j,\,u_j,\,\tau_j)$$

with $\phi(t,\,u,\,\tau) := f(t+\tau,\,\gamma(t,u,\tau))$ where

$$\gamma(t,u,\tau) \;=\; u + \tau\,f(t+\tau,\,\gamma(t,u,\tau)) \tag{1}$$

Tayler series expansion of the local error at $\tau = 0$:

$$\begin{aligned} d_\tau(t+\tau) &= u(t+\tau) - \big[u(t) + \tau\,\phi(t,u(t),\tau)\big] \\ &= u(t) + \tau\,u'(t) + \frac{\tau^2}{2}\,u''(t) + \mathcal{O}(\tau^3) - \\ &\quad - \Big[u(t) + \tau\,\big(\phi(t,u(t),0) + \tau\,\phi_\tau(t,u(t),0) + \mathcal{O}(\tau^2)\big)\Big] \end{aligned}$$

We need the partial derivative

$$\phi_\tau(t,u(t),0) \;=\; f_t\big(t,\,\gamma(t,u(t),0)\big)\cdot 1 + f_u\big(t,\,\gamma(t,u(t),0)\big)\,\gamma_\tau(t,u(t),0)$$

Value given via implicit equation:

$$\gamma(t,u,0) \;=\; u$$

For the derivative, first differentiate (1) w.r.t. τ:

$$\gamma_\tau(t,u,\tau) = f(t{+}\tau,\,\gamma(t,u,\tau)){+}\tau\big[f_t\big(t{+}\tau,\,\gamma(t,u,\tau)\big){\cdot}1{+}f_u\big(t,\gamma(t,u,\tau)\big)\gamma_\tau(t,u,\tau)\big]$$

for $\tau = 0$ this implies

$$\gamma_\tau(t,u,0) \;=\; f(t,\,\gamma(t,u,0)) \;=\; f(t,\,u)$$

Hence,

$$\begin{aligned} \phi(t,u(t),0) &= f(t,\,\gamma(t,u(t),0)) \;=\; f(t,\,u(t)) \;=\; u'(t) \\ \phi_\tau(t,u(t),0) &= f_t\big(t,\,\gamma(t,u(t),0)\big) + f_u\big(t,\,\gamma(t,u(t),0)\big)\,\gamma_\tau(t,u(t),0) \\ &= f_t(t,\,u(t)) + f_u(t,\,u(t))\,f(t,\,u(t)) \;=\; u''(t) \end{aligned}$$

and so

$$\begin{aligned} d_\tau(t+\tau) &= u(t) + \tau\,u'(t) + \frac{\tau^2}{2}\,u''(t) + \mathcal{O}(\tau^3) - \\ &\quad - \Big[u(t) + \tau\,\big(u'(t) + \tau\,u''(t) + \mathcal{O}(\tau^2)\big)\Big] \;=\; -\frac{\tau^2}{2}\,u''(t) + \mathcal{O}(\tau^3) \end{aligned}$$

Leading error term of the consistency error $\psi_\tau(u)(t)$ is $-(\tau/2)\,u''(t)$.
Consistency order 1.

2. θ-method

$$\psi_\tau(u)(t) \;=\; \Big(\frac{1}{2} - \theta\Big)\tau\, u''(t) + \mathcal{O}(\tau^2)$$

For $\theta \neq 1/2$: *Consistency order 1*
For $\theta = 1/2$ (implicit trapezoidal rule): *Consistency order 2*

3. Implicit midpoint rule

$$\psi_\tau(u)(z) \;=\; \mathcal{O}(\tau^2)$$

Consistency order 2 (1-stage method!)

In general:
An s-stage implicit Runge-Kutta method has maximal consistency order $2s$. Such methods are called Runge-Kutta methods of *Gauss type*, they are based on Gaussian quadrature rules.

Example: The implicit midpoint rule is of Gauss type.