Definition 1.48
Let $(\cdot,\cdot)$ be an inner product in $\mathbb{R}^n$ with associated norm $\|\cdot\|$.

1. $A \in \mathbb{R}^{n\times n}$ is **self-adjoint** w.r.t. $(\cdot,\cdot)$ iff
$$(A\,y,\,z) \;=\; (y,\,A\,z) \qquad \forall y, z \in \mathbb{R}^n\,.$$
2. For $(\cdot,\cdot) = (\cdot,\cdot)_{\ell^2}$ we say also **symmetric** instead of self-adjoint, because
$$A \text{ self-adjoint w.r.t. } (\cdot,\cdot)_{\ell^2} \quad\Longleftrightarrow\quad A = A^\top\,.$$
3. For $A \in \mathbb{R}^{n\times n}$ we define the **spectrum** (the finite set of eigenvalues) by
$$\sigma(A) := \big\{\lambda \in \mathbb{C} : \exists x \in \mathbb{C}^n \setminus \{0\} : A\,x = \lambda\,x\big\}.$$
If A is self-adjoint w.r.t. $(\cdot,\cdot)$, then $\sigma(A) \subset \mathbb{R}$. We define
$$\lambda_{\min}(A) := \min_{\lambda\in\sigma(A)} \lambda, \qquad \lambda_{\max}(A) := \max_{\lambda\in\sigma(A)} \lambda.$$
4. Let $A, B \in \mathbb{R}^{n\times n}$ be self-adjoint w.r.t. $(\cdot,\cdot)$
   (a) A is **positive semi-definite** ($A \geq 0$) iff $(A\,y,\,y) \geq 0 \quad \forall y \in \mathbb{R}^n$
   (b) A is **positive definite** ($A > 0$) iff $(A\,y,\,y) > 0 \quad \forall y \in \mathbb{R}^n \setminus \{0\}$
   (c) $A \geq B \;:\Longleftrightarrow\; A - B \geq 0$
   (d) $A > B \;:\Longleftrightarrow\; A - B > 0$

Lemma 1.49

(i) $A \geq 0 \quad\Longleftrightarrow\quad \forall \lambda \in \sigma(A) : \lambda \geq 0 \quad\Longleftrightarrow\quad \lambda_{\min}(A) \geq 0$

(ii) $A > 0 \quad\Longleftrightarrow\quad \forall \lambda \in \sigma(A) : \lambda > 0 \quad\Longleftrightarrow\quad \lambda_{\min}(A) > 0$

(iii) $\lambda_{\min} = \inf_{y\in\mathbb{R}^n\setminus\{0\}} \underbrace{\frac{(A\,y,\,y)}{(y,\,y)}}_{\text{Rayleigh quotient}} \qquad \lambda_{\max} = \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{(A\,y,\,y)}{(y,\,y)}$

Lemma 1.50 If A is self-adjoint and positive definite then
$$\|A\| \;=\; \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{|(A\,y,\,y)|}{(y,\,y)} \;=\; \lambda_{\max}(A), \qquad \|A^{-1}\| \;=\; \frac{1}{\lambda_{\min}(A)}\,.$$
Hence, the **condition number** $\kappa(A) := \|A\|\|A^{-1}\| = \dfrac{\lambda_{\max}}{\lambda_{\min}}$.

Lemma 1.51 Let A and C be self-adjoint w.r.t. $(\cdot,\cdot)$ and let $C > 0$. Then $C^{-1}A$ is self-adjoint w.r.t. the inner product
$$(y,\,z)_C := (C\,y,\,z)\,.$$