Definition 1.48
Let $(\cdot,\cdot)$ be an inner product in $\mathbb{R}^n$ with associated norm $\|\cdot\|$.

1. $A \in \mathbb{R}^{n\times n}$ is **self-adjoint** w.r.t. $(\cdot,\cdot)$ iff

$$(A\,y,\, z) \;=\; (y,\, A\,z) \qquad \forall y, z \in \mathbb{R}^n\,.$$

2. For $(\cdot,\cdot) = (\cdot,\cdot)_{\ell^2}$ we say also **symmetric** instead of self-adjoint, because

$$A \text{ self-adjoint w.r.t. } (\cdot,\cdot)_{\ell^2} \quad\Longleftrightarrow\quad A = A^\top\,.$$

3. For $A \in \mathbb{R}^{n\times n}$ we define the **spectrum** (the finite set of eigenvalues) by

$$\sigma(A) := \big\{\lambda \in \mathbb{C} : \exists x \in \mathbb{C}^n \setminus \{0\} : A\,x = \lambda\,x\big\}.$$

   If A is self-adjoint w.r.t. $(\cdot,\cdot)$, then $\sigma(A) \subset \mathbb{R}$. We define

$$\lambda_{\min}(A) := \min_{\lambda\in\sigma(A)} \lambda, \qquad \lambda_{\max}(A) := \max_{\lambda\in\sigma(A)} \lambda.$$

4. Let $A, B \in \mathbb{R}^{n\times n}$ be self-adjoint w.r.t. $(\cdot,\cdot)$
   (a) A is **positive semi-definite** ($A \geq 0$) iff $\quad (A\,y,\, y) \geq 0 \quad \forall y \in \mathbb{R}^n$
   (b) A is **positive definite** $\quad (A > 0)$ iff $\quad (A\,y,\, y) > 0 \quad \forall y \in \mathbb{R}^n \setminus \{0\}$
   (c) $A \geq B \;:\Longleftrightarrow\; A - B \geq 0$
   (d) $A > B \;:\Longleftrightarrow\; A - B > 0$

Lemma 1.49

(i) $A \geq 0 \quad\Longleftrightarrow\quad \forall\lambda \in \sigma(A) : \lambda \geq 0 \quad\Longleftrightarrow\quad \lambda_{\min}(A) \geq 0$

(ii) $A > 0 \quad\Longleftrightarrow\quad \forall\lambda \in \sigma(A) : \lambda > 0 \quad\Longleftrightarrow\quad \lambda_{\min}(A) > 0$

(iii) $\lambda_{\min} = \inf\limits_{y\in\mathbb{R}^n\setminus\{0\}} \underbrace{\dfrac{(A\,y,\, y)}{(y,\, y)}}_{\text{Rayleigh quotient}} \qquad \lambda_{\max} = \sup\limits_{y\in\mathbb{R}^n\setminus\{0\}} \dfrac{(A\,y,\, y)}{(y,\, y)}$

Proof: (i), (ii) clear. (iii):

$$\begin{aligned}
A \geq \alpha\, I \;&\Longleftrightarrow\; (A\,y,\, y) \geq \alpha\,(y, y) \quad \forall y \in \mathbb{R}^n \\
&\Longleftrightarrow\; \frac{(A\,y,\, y)}{(y,\, y)} \geq \alpha \quad \forall y \in \mathbb{R}^n \setminus \{0\} \\
&\Longleftrightarrow\; \inf_{y\in\mathbb{R}^n\setminus\{0\}} \frac{(A\,y,\, y)}{(y,\, y)} \geq \alpha \\
\text{and} \qquad A \geq \alpha I \;&\Longleftrightarrow\; \lambda - \alpha \geq 0 \quad \forall\lambda \in \sigma(A) \\
&\Longleftrightarrow\; \lambda_{\min}(A) \geq \alpha
\end{aligned}$$

$\square$

Lemma 1.50 If A is self-adjoint and positive definite then

$$\|A\| \;=\; \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{|(A\,y,\,y)|}{(y,\,y)} \;=\; \lambda_{\max}(A), \qquad \|A^{-1}\| \;=\; \frac{1}{\lambda_{\min}(A)}\,.$$

Hence, the **condition number** $\kappa(A) := \|A\|\|A^{-1}\| = \dfrac{\lambda_{\max}}{\lambda_{\min}}$.

Proof: For a self-adjoint operator,

$$\|y\| \;=\; \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{|(A\,y,\,y)|}{(y,\,y)}$$

(see lectures/books on Functional Analysis).
Since A is positive definite, $|(A\,y,\,y)| = (A\,y,\,y)$ and thanks to Lemma 1.49(iii),

$$\|A\| = \lambda_{\max}(A).$$

First of all A^{-1} exists because of the positive definiteness. Since

$$A\,x = \lambda\,x \iff \frac{1}{\lambda}\,(A\,x) = A^{-1}\,(A\,x),$$

we see that $\lambda \in \sigma(A) \iff 1/\lambda \in \sigma(A^{-1})$. It is also easy to see that A^{-1} is self-adjoint with respect to $(\cdot,\cdot)$, and so

$$\|A^{-1}\| \;=\; \lambda_{\max}(A^{-1}) \;=\; \frac{1}{\lambda_{\min}(A)} \;=\; \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{(y,y)}{(A^{-1}y,\,y)}.$$

□

Lemma 1.51 Let A and C be self-adjoint w.r.t. $(\cdot,\cdot)$ and let $C > 0$.
Then $C^{-1}A$ is self-adjoint w.r.t. the inner product

$$(y,\,z)_C := (C\,y,\,z)\,.$$

Proof:

$$\begin{aligned}(C^{-1}A\,y,\,z)_C \;&=\; (C\,C^{-1}\,A\,y,\,z) \;=\; (A\,y,\,z) \;=\; (y,\,A\,z)\\ &=\; (y,\,C\,C^{-1}A\,z) \;=\; (C\,y,\,C^{-1}A\,z) \;=\; (y,\,C^{-1}A\,z)_C\end{aligned}$$

□