Lemma 1.63

In the k-th step of the CG method assume that $r_{k-1} \neq 0$. Then

(a) $p_{k-1} \neq 0$

(b) $\mathcal{K}_k(A, r_0) \; = \; \operatorname{span}(r_0, \ldots, r_{k-1}) \; = \; \operatorname{span}(p_0, \ldots, p_{k-1})$

(c) $\forall j = 0, 1, \ldots, k-1 : \quad (r_k, p_j) \; = \; 0$

(d) $\forall j = 0, 1, \ldots, k-1 : \quad (p_k, p_j)_A \; = \; 0$

Proof by induction.

For $k = 1$ the statements are trivial or follow from (20), (21).

Suppose that (a)–(d) hold for k and assume that $r_k \neq 0$.

$$r_k \; = \; r_{k-1} - \alpha_k \, A \, p_{k-1} \; \in \; \mathcal{K}_k(A, r_0) + A\big(\mathcal{K}_k(A, r_0)\big) \; \subset \; \mathcal{K}_{k+1}(A, r_0).$$

Proof of (a) and (b).

From (b) and (c) we know that $r_k \perp \mathcal{K}_k(A, r_0)$, which implies that

$$\mathcal{K}_k(A, r_0) \; \subsetneq \; \operatorname{span}(r_0, \ldots, r_k) \; \subset \; \mathcal{K}_{k+1}(A, r_0).$$

However, $\dim(\mathcal{K}_{k+1}(A, r_0)) = \dim(\mathcal{K}_k(A, r_0)) + 1$. Therefore,

$$\operatorname{span}(r_0, \ldots, r_k) \; = \; \mathcal{K}_{k+1}(A, r_0).$$

From the definition of the algorithm we know that $r_k = p_k - \beta_{k-1} \, p_{k-1}$. Hence,

$$\operatorname{span}(p_0, \ldots, p_k) \; = \; \operatorname{span}(p_0, \ldots, p_{k-1}, r_k) \; \overset{\text{(b)}}{=} \; \operatorname{span}(r_0, \ldots, r_k).$$

This means, (a) and (b) hold for $k+1$.

Proof of (c).

From formula (20) we know that $(r_{k+1}, p_j) = 0$ for $j = k$. For $j < k$,

$$(r_{k+1}, p_j) \; = \; (r_k - \alpha_k \, A \, p_k, p_j) \; = \; (r_k, p_j) - \alpha_k (A \, p_k, p_j) \; \overset{\text{(c),(d)}}{=} \; 0.$$

Thus, $r_{k+1} \perp \operatorname{span}(p_0, \ldots, p_k) = \mathcal{K}_{k+1}(A, r_0)$ and so (c) holds for $k+1$.

Proof of (d).

From formula (21) we know that $(A \, p_{k+1}, p_j) = 0$ for $j = k$. For $j < k$,

$$(A \, p_{k+1}, p_j) \; = \; (p_{k+1}, A \, p_j) \; = \; \underbrace{(r_{k+1}, \underbrace{A \, p_j}_{\in \mathcal{K}_{k+1}(A, r_0)})}_{=0} + \beta_k \underbrace{(p_k, A \, p_j)}_{\overset{\text{(d)}}{=} 0} \; = \; 0.$$

This means (d) holds for $k+1$, which completes the proof. □