Theorem 1.27 (Banach's fixed point theorem)

Let W be a closed subset of a Banach space V (with norm $\|\cdot\|$) and $G : W \to W$ a contraction, i.e., there exists a constant $q < 1$ with

$$\|G(v) - G(w)\| \;\leq\; \|v - w\| \qquad \forall v,\, w \in W.$$

Then there exists a unique element $u \in W$ with

$$u \;=\; G(u).$$

The sequence $(u_k)_{k\in\mathbb{N}_0}$, given by the fixed point iteration

$$u_{k+1} \;=\; G(u_k)$$

converges to the solution u for arbitrary initial values $u_0 \in W$. We also have the following error estimates for all $k \in \mathbb{N}_0$:

$$\|u_{k+1} - u\| \;\leq\; q\,\|u_k - u\| \qquad \text{(q-linear convergence)}$$

$$\|u_{k+1} - u\| \;\leq\; q^{k+1}\,\|u_0 - u\| \qquad \text{(r-linear convergence)}$$

$$\|u_{k+1} - u\| \;\leq\; \frac{q^{k+1}}{1-q}\,\|u_1 - u_0\| \qquad \text{(constructive a-priori error estimate)}$$

$$\|u_{k+1} - u\| \;\leq\; \frac{q}{1-q}\,\|u_{k+1} - u_k\| \qquad \text{(constructive a-posteriori error estimate).}$$

Proof: Let $u_0 \in W$ be arbitrary but fixed.
First, we show that $(u_k)_{k\in\mathbb{N}_0}$ is a Cauchy sequence. For all $m \in \mathbb{N}$:

$$\|u_{m+1} - u_m\| \;=\; \|G(u_m) - G(u_{m-1})\| \;\leq\; q\,\|u_m - u_{m-1}\|\,.$$

This implies that for all $m \in \mathbb{N}_0$:

$$\|u_{m+1} - u_m\| \;\leq\; q^m\,\|u_1 - u_0\|\,.$$

Using the triangle inequality and the estimate above, we obtain that for $k < \ell$:

$$\|u_\ell - u_k\| \leq \sum_{i=k}^{\ell-1} \|u_{i+1} - u_i\| \leq \sum_{i=k}^{\ell-1} q^i\,\|u_1 - u_0\| \leq \Big(\sum_{i=k}^{\infty} q^i\Big)\,\|u_1 - u_0\| = \frac{q^k}{1-q}\,\|u_1 - u_0\|.$$

Since $0 \leq q < 1$ and q^k can be made arbitrary small for large enough k, $(u_k)_{k\in\mathbb{N}_0}$ is indeed a Cauchy sequence. Since V is complete and W is closed, we know that there exists a limit function $u \in W$: $u_k \overset{k\to\infty}{\to} u$.
As a contraction, G is Lipschity continuous, and thus continuous. Hence,

$$u \;=\; \lim_{k\to\infty} u_k \;=\; \lim_{k\to\infty} G(u_{k-1}) \;=\; G\big(\lim_{k\to\infty} u_{k-1}\big) \;=\; G(u).$$

To see that the fixed point equation has a unique solution, we assume that there are two solutions u_1, u_2, i.e. $u_1 = G(u_1)$ and $u_2 = G(u_2)$. Then,

$$\|u_2 - u_1\| \;=\; \|G(u_2) - G(u_1)\| \;\leq\; q\,\|u_2 - u_1\|.$$

Since $0 \leq q < 1$, this is only possible if $\|u_2 - u_1\| = 0$, which means that $u_1 = u_2$.
The first and second error estimate follow immediately from the contraction property. The third one follows from $\|u_\ell - u_k\| \leq q^k/(1-q)\,\|u_1 - u_0\|$ (see above) by sending $\ell \to \infty$. For the fourth estimate, we need to see that $\|u_{k+1} - u_{\ell+1}\| \leq q\,\sum_{i=k}^{\ell-1} q^{i-k}\,\|u_{k+1} - u_k\|$ (using the contraction property and the triangle inequality) and then send $\ell \to \infty$. $\square$