$\boxed{74}$ Consider the classical Runge-Kutta method of order 4,

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Show that this method has the stability function

$$R(z) \;=\; 1 + z + \frac{1}{2}\,z^2 + \frac{1}{6}\,z^3 + \frac{1}{24}\,z^4$$

and that $e^z - R(z) = \mathcal{O}(z^5)$ as $z \to 0$.

$\boxed{75}$ Complete the proof of Lemma 2.42 (consistency analysis of the θ-method): Show that

$$\Big|\Big(u(t_{j+1}) - u(t_j) - \tau_j\,\big[(1-\theta)\,u'(t_j) + \theta\,u'(t_{j+1})\big], v\Big)\Big| \;\le\; \tau_j \int_{t_j}^{t_{j+1}} \|u''(\sigma)\|\,d\sigma\,\|v\|$$

Programming (in C, C^{++}, `matlab`, or whatever you like):

Consider the problem to find $u : [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} u'(t) &= -50\,u(t) \qquad \forall t \in (0,T), \\ u(0) &= 1, \end{aligned}$$

with $T = 1$. Find the exact solution.

$\boxed{76,\ 77}$ Implement the *explicit* Euler method for the above problem with constant time steps $\tau_j = \tau$. Run it for $\tau = 1/60$, $1/30$, $1/26$, $1/24$, and $1/20$. For each run, plot the numerical solution and the exact solution.

$\boxed{78,\ 79}$ Implement the *implicit* Euler method for the above problem with constant time steps $\tau_j = \tau$. Run it for $\tau = 1/60$, $1/30$, $1/26$, $1/24$, and $1/20$. For each run, plot the numerical solution and the exact solution.