**68** Assume that $f$ fulfills the Lipschitz condition (13), i.e.

$$\|f(t,v) - f(t,w)\| \;\leq\; L\,\|v-w\| \qquad \forall t \in [0,T] \quad \forall v,w \in \mathbb{R}^n\,.$$

Furthremore, let the sequences $(u_j)$ and $(v_j)$ be given according to the (perturbed) explicit Euler method:

$$\left.\begin{aligned} u_{j+1} &= u_j + \tau_j\, f(t_j,u_j) \\ v_{j+1} &= v_j + \tau_j\,[f(t_j,v_j) + y_{j+1}] \end{aligned}\right\} \qquad \forall j = 0,\dots,m-1,$$

and $v_0 = u_0 + y_0$ for given (but arbitrary) values $u_0$ and $y_0, \dots, y_m \in \mathbb{R}^n$. Show that then,

$$\|u_j - v_j\| \;\leq\; e^{(t_j-t_0)L}\,\|y_0\| + \frac{1}{L}\left(e^{(t_j-t_0)L} - 1\right) \max_{k=1,\dots,j} \|y_k\|$$

for all $\tau > 0$.
*Hint:* Show and use $e^{(t_j-t_k)L}\,\tau_{k-1} \leq \int_{t_{k-1}}^{t_k} e^{(t_j-s)L}\,ds$.

**69** Recall that the definitions of consistency, stability, and convergence depend on the norms $\|\cdot\|_{X_\tau}$ and $\|\cdot\|_{Y_\tau}$. In this exercise, we replace $\|\cdot\|_{Y_\tau}$ by $\|\cdot\|_{X_\tau}$.

Use Exercise **68** to show an estimate of the form

$$\|e_\tau\|_{X_\tau} \;\leq\; C\,\|\psi_\tau(u)\|_{X_\tau}$$

for the explicit Euler method with a stablitity constant $C$ independent of $\tau$.
Furthermore, show that for exact solutions $u \in C^2([0,T],\mathbb{R}^n)$,

$$\|\psi_\tau(u)\|_{X_\tau} \;\leq\; K\,\tau$$

with $K = \max\limits_{s\in[0,T]} \|u''(s)\|$.

**70** Consider the general explicit 2-stage Runge-Kutta method

$$\begin{aligned} g_1 &= u_j \\ g_2 &= u_j + \tau_j\, a_{2,1}\, f(t_j,\, g_1) \\ u_{j+1} &= u_j + \tau_j\,\big[b_1\, f(t_j,\, g_1) + b_2\, f(t_j + c_2\,\tau_j,\, g_2)\big] \end{aligned}$$

with coefficients $a_{2,1}$, $b_1$, $b_2$ and $c_2$ for the approximate solution of the initial value problem to find $u : [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} u'(t) &= f(t,\, u(t)) \qquad \forall t \in (0,T), \\ u(t) &= u_0\,, \end{aligned}$$

where $u_0 \in \mathbb{R}$ is given and $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is sufficiently smooth. Provide a Taylor series expansion of the *local error* of the form

$$d_\tau(t+\tau) \;=\; A_0 + A_1\,\tau + A_2\,\tau^2 + A_3\,\tau^3 + \mathcal{O}(\tau^4),$$

with the expressions $A_0,\dots,A_3$ depending only on $a_{2,1}$, $b_1$, $b_2$, $c_2$, $f$, and its derivatives, but not on $\tau$.

**71** Continue exercise **70** and find necessary conditions on the coefficients $a_{2,1}$, $b_1$, $b_2$, and $c_2$ such that the consistency order of the method is at least 2, i.e. such that for all sufficiently smooth functions $f$,

$$A_0 = A_1 = A_2 = 0.$$

Is it possible to get consistency order 3?

In the following, let $X$ be a Banach space with norm $\|\cdot\|$ and consider the general initial value problem to find $u : [0,\infty) \to X$ such that

$$\begin{aligned} u'(t) &= f(t, u(t)) \qquad \forall t \geq 0, \\ u(0) &= u_0\,, \end{aligned}$$

with $u_0 \in X$ and $f : [0,\infty) \times X \to X$ given.

**72** Assume that there exists a constant $L > 0$ such that

$$\|f(t,v) - f(t,w)\| \leq L\,\|v - w\| \qquad \forall t \geq 0 \qquad \forall v, w \in X. \tag{12.1}$$

Show that for each given $t_{j+1} > 0$ and $u_j \in X$, there exists a unique solution $\boldsymbol{v}$ to the implicit equation

$$\boldsymbol{v} = u_j + \tau_j\, f(t_{j+1}, \boldsymbol{v}), \tag{12.2}$$

if $0 < \tau_j < 1/L$. *Hint:* use Banach's fixed point theorem.

**73** Assume that $X$ is a Hilbert space with the inner product $(\cdot,\cdot)$ and that

$$\big(f(t,v) - f(t,w),\, v - w\big) \leq 0 \qquad \forall t \geq 0 \qquad \forall v, w \in X \tag{12.3}$$

holds additionally to (12.1). Show that for each given $\tau_j > 0$, $t_{j+1}$ and $u_j \in X$, there exists a unique solution $\boldsymbol{v}$ to the implicit equation (12.2).
*Hint:* apply Banach's fixed point theorem to the equivalent equation

$$\boldsymbol{v} = G(\boldsymbol{v}) := (1-\rho)\boldsymbol{v} + \rho\big(u_j + \tau_j\, f(t_{j+1}, \boldsymbol{v})\big),$$

where you have to choose the parameter $\rho \in (0,1)$ such that $G$ is a contraction.