Assume the notations of Tutorial 10 and assume additionally that

- $a(\cdot,\cdot)$ is bounded and coercive with coercivity constant μ_1
- all the functions $t \mapsto \langle F(t),\, v\rangle$ are continuous with respect to t and that the solution $u \in C^1([0,T],\, V)$.

In the following $u_h \in C^1([0,T], V_h)$ is the solution of

$$\begin{aligned} (u_h'(t),\, v_h)_H + a(u_h(t),\, v_h) &= \langle F(t),\, v_h\rangle \qquad \forall v_h \in V_h \quad \forall t \in (0,T), & (11.1)\\ (u_h(0),\, v_h)_H &= (u_0,\, v_h)_H \qquad \forall v_h \in V_h\,. & (11.2) \end{aligned}$$

Recall the projectors $R_h : V \to V_h$ and $P_h : V \to V_h$ defined by

$$\begin{aligned} a(R_h w,\, v_h) &= a(w,\, v_h) \qquad \forall v_h \in V_h,\\ (P_h w,\, v_h)_H &= (w,\, v_h)_H \qquad \forall v_h \in V_h, \end{aligned}$$

where $w \in V$.

62 Show that

$$\frac{1}{2}\,\|u_h(T)\|_H^2 + \int_0^T a(u_h(t),\, u_h(t)) \;=\; \frac{1}{2}\,\|u_h(0)\|_H^2 + \int_0^T \big\langle F(t),\, u_h(t)\big\rangle\, dt. \qquad (11.3)$$

Hint: Choose $v_h = u_h(t)$ in (11.1) and use Exercise **59**.

63 Show that

$$\|P_h v\|_H \;\le\; \|v\|_H \qquad \forall v \in V,$$

and that $\|u_h(0)\|_H \le \|u_0\|_H$. Show also that if $a(\cdot,\cdot)$ is symmetric then

$$\|R_h v\|_a \;\le\; \|v\|_a \qquad \forall v \in V,$$

where $\|v\|_a := \sqrt{a(v,\, v)}$.

64 Show the following a-priori bound for u_h:

$$\|u_h\|_{L^2((0,T),V)} \;\le\; \frac{1}{2\mu_1}\Big(\|F\|_{L^2((0,T),V^*)} + \sqrt{\|F\|^2_{L^2((0,T),V^*)} + 2\mu_1\,\|u_0\|_H^2}\Big).$$

Hint: Bound the left hand side of (11.3) from below and the right hand side of (11.3) from above to show that

$$\|u_h\|^2_{L^2((0,T),V)} \;\le\; \frac{1}{2\,\mu_1}\,\|u_h(0)\|_H^2 + \frac{1}{\mu_1}\,\|F\|_{L^2((0,T),V^*)}\,\|u_h\|_{L^2((0,T),V)}\,,$$

and use Exercise **63**.

65 Show that

$$\frac{d}{dt}\|\theta_h(t)\|_H \;\le\; \|\rho_h'(t)\|_H - \frac{\mu_1}{c^2}\,\|\theta_h(t)\|_H \qquad \forall t \in (0,\, T) \text{ a.e.}$$

Hint: See (and modify) the proof of Lemma 2.12.

66 Show that

$$\|\theta_h(t)\|_H \leq e^{-\mu_1 c^{-2} t} \|\theta_h(0)\|_H + \int_0^t e^{-\mu_1 c^{-2}(t-s)} \|\rho_h'(s)\|_H \, ds.$$

Hint: Estimate the term

$$\frac{d}{ds}\left[e^{\mu_1 c^{-2} s} \|\theta_h(s)\|_H\right]$$

using Exercise 65 and integrate over $[0, t]$ w.r.t. s.

67 Let $H = L^2(0,1)$, $V = \{v \in H^1(0,1) : v(0) = 0\}$, and V_h the corresponding Courant FE space. Show that there exists a positive constant C independent of h such that

$$\|[I - P_h]w\|_{L^2(0,1)} \leq C\, h^2 \, |w|_{H^2(0,1)} \qquad \forall w \in H^2(0,1).$$

Hint: Use Céa's Lemma in $L^2(0,1)$.