Programming

$\boxed{50}$ Implement the remaining routines of `MDSPreconditioner` according to `mds.hh`.

$\boxed{51}$ Implement the PCG method (with the MDS preconditioner).
Hint: Copy and change your existing CG code according to the lecture (see slide "Preconditioner Conjugate Gradient Method (PCG)" `pcg.pdf`)
The new routine should look like

```
template <class PRECONDITIONER, class MATRIX, class VECTOR, class REAL>
int
PCG (const MATRIX & A, VECTOR & x, const VECTOR & b,
     const PRECONDITIONER & C, int & max_iter, REAL & tol);
```

or

```
int PCG (const SMatrix& A, Vector& x, const Vector& b,
         const MDSPreconditioner& C, int& max_iter, double& tol);
```

$\boxed{52}$ Solve a boundary value problem of your choice (e.g. one from Exercise $\boxed{33}$) with the PCG-preconditioned PCG method. Start with a simple mesh of e.g. two elements and perform uniform refinement. The core part of your main program could be as follows:

create `mesh` with two elements
create `K` and `f` from `mesh` (with BC!)
call `mds.InitDiagonal (0, K)`
for `m`$=1,\dots,L-1$
  call `mesh.RefineUniform()`
  create `K` and `f` from `mesh` (with BC!)
  call `mds.InitDiagonal (m, K)`
end for
call `PCG`

Report the number of PCG iterations for 2^k levels, where $k = 0, 1, \dots, 10$.

$\boxed{53}$ Let the assumptions of Lemma 1.73 (nonlinear Lax-Milgram) hold, i.e., let V be a Hilbert space, $V_0 \subset V$ a closed subspace, $g \in V$, $V_g := g + V_0$, and let $A : V \to V_0^*$ be a nonlinear operator which is strongly monotone on V_g

$$\exists \mu_1 > 0 : \qquad \langle A(v) - A(w), v - w \rangle \;\geq\; \mu_1 \, \|v - w\|_V^2 \qquad \forall v, w \in V_0$$

and Lipschitz continuous, i.e.

$$\exists \mu_2 \geq 0 : \qquad \|A(v) - A(w)\|_{V_0^*} \;\leq\; \mu_2 \, \|v - w\|_V \qquad \forall v, w \in V_0 \,,$$

where $\|F\|_{V_0^*} := \sup_{v \in V_0 \setminus \{0\}} \frac{|\langle F, v \rangle|}{\|v\|_V}$.

Let $F_1, F_2 \in V_0^*$ and let $u_1 \in V_g$: $A(u_1) = F_1$, $u_2 \in V_g : A(u_2) = F_2$. Show that there exists a constant $C \geq 0$:

$$\|u_1 - u_2\|_V \;\leq\; C \, \|F_1 - F_2\|_{V_0^*} \,,$$

i.e. the solution depends *continuously* on the data.

The following two exercises consider Example 1.71. Let $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and bounded function. For $V := H^1(0, 1)$ and $V_0 := \{v \in H^1(0, 1) : v(0) = 0\}$, we define the operator $A : V \to V_0^*$ by the relation

$$\langle A(u), v \rangle \;=\; \int_0^1 \alpha(|u'(x)|)\, u'(x)\, v'(x)\, dx$$

and set $V_g := \{v \in H^1(0, 1) : v(0) = g_0\}$.

Note: $A(u)$ is well-defined because the function $x \mapsto \alpha(|u'(x)|)$ is Lebesgue-measurable in $(0, 1)$, the function α is bounded, so $\alpha(|u'|)u' \in L^2(0, 1)$, and therefore the above integral is always finite.

$\boxed{54}$ Assume that there exists $M \geq 0$:

$$\big|\alpha(s)s - \alpha(t)t\big| \;\leq\; M\,|s - t| \qquad \forall s,\, t \in \mathbb{R}_0^+\,.$$

Show that then $A : V \to V_0^*$ is Lipschitz continuous on V_g with Lipschitz constant $\mu_2 = 3M$.

Hint: Use that for $t \geq 0$:

$$\begin{aligned} \alpha(|s|)s - \alpha(|t|)t \;&=\; \alpha(|s|)(s-t) + \big[\alpha(|s|) - \alpha(|t|)\big]t \\ &=\; \alpha(|s|)(s-t) + \alpha(|s|)\,\big(|t| - |s|\big) + \alpha(|s|)|s| - \alpha(|t|)|t| \end{aligned}$$

Show and use that $\alpha(|s|) \leq M$ for all $s \in \mathbb{R}$.

$\boxed{55}$ Assume that there exists $m > 0$:

$$\alpha(s)s - \alpha(t)t \;\geq\; m\,(s-t) \qquad \forall s \geq t \geq 0.$$

Show that then $A : V \to V_0^*$ is strongly monotone on V_g where μ_1 depends on m and the constant C_F from Friedrichs' inequality.

Hint: Set $\alpha_2(s) := \alpha(s) - m$. Show and use that for all $s,\, t \in \mathbb{R}$:

(a) $\quad \alpha_2(|s|) \;\geq\; 0$

(b) $$\begin{aligned} (\alpha(|s|)s - \alpha(|t|)t)(s-t) \;&=\; \big[\alpha_2(|s|)s - \alpha_2(|t|)t\big](s-t) + m\,(s-t)^2 \\ &\geq\; \big[\alpha_2(|s|)|s| - \alpha_2(|t|)|t|\big](|s| - |t|) + m\,(s-t)^2 \end{aligned}$$

(c) $\quad \big[\alpha_2(|s|)|s| - \alpha_2(|t|)|t|\big](|s| - |t|) \;\geq\; 0$