$\boxed{38}$ Let A be self-adjoint and positive definite with respect to $(\cdot,\cdot)$, and $b \in \mathbb{R}^n$. Recall the energy functional

$$J_A(y) := \tfrac{1}{2}(A\,y,\,y) - (b,\,y).$$

Let $x_k \in \mathbb{R}^n$ be a given approximation and $p_k \in \mathbb{R}^n$ a given search direction. Compute

$$\alpha_k \;=\; \operatorname*{argmin}_{\alpha\in\mathbb{R}} J_A(x_k + \alpha\,p_k),$$

in terms of p_k, r_k, and A (where $r_k := b - A\,x_k$).

$\boxed{39}$ Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that for $M_\tau = I - \tau\,A$, $\tau \in \mathbb{R}$:

$$\|M_\tau\|_{\ell_2} \;=\; \max_{\lambda\in\sigma(A)} |1-\tau\,\lambda| \;=\; q(\tau)$$

with $q(\tau) = \max\big(|1-\tau\,\lambda_{\max}(A)|,\,|1-\tau\,\lambda_{\min}(A)|\big)$.

$\boxed{40}$ Let $A \in \mathbb{R}^{n\times n}$ be a symmetric indefinite matrix with at least one positive and one negative eigenvalue. Show that

$$\max_{\lambda\in\sigma(A)} |1-\tau\,\lambda| \;>\; 1 \qquad \forall \tau \neq 0.$$

$\boxed{41}$ Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $\lambda_- < 0$ and $\lambda_+ > 0$ be two eigenvalues with their corresponding eigenvectors e_- and e_+. Show that there is *no* parameter $\tau \in \mathbb{R}$ such that Richardson's method

$$x_{k+1} \;=\; x_k + \tau(b - A\,x_k)$$

converges for the initial vector $x_0 = x + e_- + e_+$, where $x = A^{-1}\,b$.

$\boxed{42}$ We have seen in the lecture that for CG,

$$x_k \in x_0 + \mathcal{K}_k(A,\,r_0) \qquad \text{and} \qquad r_k \perp \mathcal{K}_k(A,\,r_0),$$

where $\perp$ means orthogonal with respect to the (given) inner product.

Consider now the GMRES (generalized minimal residual) method, which can also be applied to indefinite matrices. There the iterates are constructed such that

$$x_k \in x_0 + \mathcal{K}_k(A,\,r_0) \qquad \text{and} \qquad \|b - A\,x_k\|_{\ell^2} = \min_{y\in x_0+\mathcal{K}_k(A,\,r_0)} \|b - A\,y\|_{\ell^2}\,.$$

Show that in this case, for $A \in \mathbb{R}^n$ (not necessarily symmetric positive definite),

$$r_k \perp A(\mathcal{K}_k(A,\,r_0)),$$

where here, $\perp$ means orthogonal in the Euclidean inner product.
Hint: Rewrite the above minimization problem as an equivalent variational problem.

Programming

$\boxed{43}$ Solve a boundary value problem of your choice with the conjugate gradient method. Use `cg.hh` from the webpage and perform the necessary modifications.

Report the number of iterations for 2^k elements, $k = 4,\ldots,10$. Compare to the number of iterations needed by Richardson's method (Exercise $\boxed{35}$) and the theoretical results from the lecture.