Programming:

$\boxed{32}$ Write a function

```
double ApproxL2Error (const Mesh& mesh, const Vector& uh,
                      RealFunction sol);
```

which approximates the error $\|u - u_h\|_{L^2(0,1)}$ (where uh=$\underline{u}_h$ and sol=u) using the *midpoint rule* on each element:

$$\|u - u_h\|^2_{L^2(0,1)} = \sum_{k=1}^{n_h} \int_{T_k} |u - u_h|^2 \, dx \approx \sum_{k=1}^{n_h} h_k \, |u(x_k^*) - u_h(x_k^*)|^2 ,$$

where $x_k^* := \frac{1}{2}(x_{k-1} + x_k)$ is the midpoint of element T_k and $h_k = x_k - x_{k-1}$.

$\boxed{33}$

- If your student ID (Matrikelnummer) is *even* (dividable by 2) consider the boundary value problem

$$-u''(x) = 2\,x + 1, \qquad u(0) = 3, \qquad u'(1) = -1/2.$$

  Find the *exact* solution u of this equation – it must be a cubic polynomial!

- If your student ID is *odd* (not divitable by 2) consider the exact solution

$$u(x) = \sin\Big(\frac{5\,\pi}{2}\big(x - \frac{1}{5}\big)\Big),$$

  calculate $f(x) = -u''(x)$, $g_0 = u(0)$, and $g_1 = u'(1)$ and solve the corresponding boundary value problem numerically.

Implement the exact solution as a function `double mySolution (double x);` and use `ApproxL2Error` to approximate the errors $\|u - u_h\|_{L^2(0,1)}$ for a series of equidistant meshes with 32, 64, 128, 256, 512, and 1024 elements. Report h and the error. How do the results relate to the theory you heard in the lecture?

$\boxed{34}$ Implement a function

```
void SMatrix :: Mult (const Vector& v, Vector& r) const;
```

in your matrix class, which computes r=$K_h\, v$ for v=v.

$\boxed{35}$ Solve a boundary value problem of your choice with Richardson's iteration instead of the Thomas algorithm (exercise $\boxed{31}$). Use `richardson.hh` from the webpage and perform the necessary modifications.

How do you have to choose τ in order to make the iteration converge?

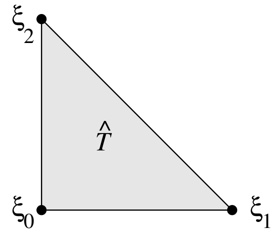
Report the number of iterations for 2, 4, 8, 16, 32, and 64 elements. How do the results relate to the theory you heard in the lecture?

please turn over

$\boxed{36}$ Let $\widehat{T} := \{(x,\, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x + y \leq 1\}$ denote the two-dimensional reference element with the corner points $\xi_0 = (0,\, 0)$, $\xi_1 = (1,\, 0)$, and $\xi_2 = (0,\, 1)$. Let $\widehat{\varphi}_0$, $\widehat{\varphi}_1$, and $\widehat{\varphi}_2$ denote affine linear functions on $\widehat{T}$ that fulfill

$$\widehat{\varphi}_i(\xi_j) = \delta_{ij} \qquad \forall i,\, j \in \{0,\, 1,\, 2\}.$$

Derive an explicit formula for $\widehat{\varphi}_0$, $\widehat{\varphi}_1$, and $\widehat{\varphi}_2$ in terms of $\xi = (\xi^{(1)},\, \xi^{(2)})$.



$\boxed{37}$ Let $\mathcal{T}_h$ be an equidistant mesh of $(0, 1)$, let V_{0h} be the space of continuous piecewise affine linear functions that vanish at 0, and let K_h be the stiffness matrix corresponding to the 1D model problem. Show that there exists a constant $C > 0$ independent of h such that

$$\kappa(K_h) \ \geq \ C\, h^{-2}$$

Hint:
Use the Rayleigh quotient for the *special* vector $\underline{v}_h = (1, 0, \ldots, 0)^\top$ to obtain a lower bound for $\lambda_{\max}(K_h)$.
For an upper bound of $\lambda_{\min}(K_h)$ use the vector $\underline{v}_h = (h, 2h, 3h, \ldots, 1)^\top$.