$\boxed{13}$ Consider our 1D model problem from the lecture with $g_0 = 0$, i.e.

$$\text{Find } u \in V_0 : \quad \int_0^1 u'\, v'\, dx \;=\; \int_0^1 f\, v\, dx + g_1\, v(1) \qquad \forall v \in V_0\,,$$

where $V_0 = \{v \in H^1(0,1) : v(0) = 0\}$.

Specify precisely the boundary value problem for $w_k \in V_0$ which corresponds to

$$w_k \;=\; \mathcal{R}(F - A\, u),$$

where $\mathcal{R} : V_0^* \to V_0$ is the Riesz isomorphism.

$\boxed{14}$ Prove Lemma 1.30 from the lecture: Let V be a Hilbert space, $a(\cdot,\, \cdot) : V \times V \to \mathbb{R}$ a bilinear form that is symmetric and non-negative on V. Let $V_0 \subset V$ and $V_g = g + V_0$ with $g \in V$. Then

$$\text{find } u \in V_g : \quad a(u,\, v) = \langle F,\, v\rangle \quad \forall v \in V_0$$

is equivalent to

$$\text{find } u \in V_g : \quad J_a(u) = \inf_{v \in V_g} J_a(v),$$

with $J_a(v) = \frac{1}{2} a(v,\, v) - \langle F,\, v\rangle$.
Hint: Follow the proof of Lemma 1.24.

$\boxed{15}$ Show that

$$a(v_h,\, w_h) \;=\; (K_h\, \underline{v}_h,\, \underline{w}_h)_{\ell^2}$$

where K_h is the stiffness matrix (defined according to the lecture), $v_h,\, w_h \in V_{0h}$ (arbitrary), $\underline{v}_h$, $\underline{w}_h$ are the corresponding vectors according to the Ritz isomorphism, and $(\cdot, \cdot)_{\ell^2}$ is the Euclidean inner product. Show also that for the load vector $\underline{f}_h$,

$$\langle \widehat{F},\, v_h\rangle \;=\; (\underline{f}_h,\, \underline{v}_h)_{\ell^2}\,.$$

$\boxed{16}$ Consider the *pure Neumann problem* with homogeneous boundary conditions.
Find $u \in V_0 = V = H^1(0,1)$:

$$\int_0^1 u'(x)\, v'(x)\, dx \;=\; \int_0^1 f(x)\, v(x)\, dx \qquad \forall v \in V.$$

Let $\mathcal{T}_h = \{T_1, \ldots, T_{n_h}\}$ be a mesh of $(0,1)$ and let

$$V_{0h} = V_h := \left\{v \in C[0,1] : v_{|T} \in \mathcal{P}^1 \quad \forall T \in \mathcal{T}_h\right\}$$

be the space of continuous and piecewise affine linear functions. Write the corresponding stiffness matrix $K_h \in \mathbb{R}^{(n_h+1)\times(n_h+1)}$ in terms of element stiffness matrices:

$$(K_h\, \underline{v}_h,\, \underline{w}_h)_{\ell^2} \;=\; \sum_{k=1}^{n_h} \left(K_h^{(k)} \begin{pmatrix} v_{k-1} \\ v_k \end{pmatrix},\, \begin{pmatrix} w_{k-1} \\ w_k \end{pmatrix} \right)_{\ell^2}$$

Write the corresponding load vector $\underline{f}_h \in \mathbb{R}^{n_h+1}$ in terms of element load vectors:

$$(\underline{f}_h,\, \underline{v}_h)_{\ell^2} \;=\; \sum_{k=1}^{n_h} \left(f_h^{(k)},\, \begin{pmatrix} v_{k-1} \\ v_k \end{pmatrix} \right)_{\ell^2}$$

Specify $K_h^{(k)}$ and $f_h^{(k)}$.

Programming.
In C^{++} (or C) only! (no Fortran, no Java, no matlab)
You will need a C/C^{++} compiler and an editor, or an integrated development environment (like DevC^{++}, eclipse, Visual Studio, ...).

The following two exercises are a "warm-up" and will prepare for following programming excercies.

$\boxed{17}$ Use

```
typedef double Vec2[2];
typedef double Mat22[2][2];
```

to define a vector type `Vec2` in $\mathbb{R}^2$ and a 2×2 matrix type `Mat22`.

Write a function

```
void mult (const Mat22& mat, const Vec2& vec, Vec2& res);
```

that calculates the matrix-vector product. If `mat`$=M$, `vec`$=\underline{v}$, and `res`$=\underline{r}$, then $\underline{r} = M\,\underline{v}$.

Test `mult` for at least two examples.

$\boxed{18}$ Use

```
typedef double (*RealFunction)(double x);
```

to define a function type `RealFunction`. Write a function

```
void printValues (RealFunction f, double a, double b);
```

that evaluates `f` at `a` and `b` and prints the values on the screen.

Test `printValues` with your own favourite function

```
double myFunc (double x)
{
  return ...
}
```

Provide your code on a USB-stick or e-mail it to me not later than Thursday 1.30 am.