07 Show *Poincaré's inequality*: There exists a constant $C_P > 0$ such that

$$\|v\|_{L_2(0,1)} \;\leq\; C_P\Big\{\Big(|v|^2_{H^1(0,1)} + \int_0^1 v(x)\,dx\Big)^2\Big\}^{1/2} \qquad \forall v \in H^1(0,1)\,.$$

Hint: Integrate the identity

$$v(y) \;=\; v(x) + \int_x^y v'(z)\,dz$$

over the whole interval $(0,\,1)$ with respect to x. The rest of the proof is then similar to the one of Friedrichs' inequality (see your lecture notes).

08 *The pure Neumann problem (continuation of exercise* **06***).*
Show that weak formulation of the pure Neumann problem has a solution if and only if $\forall c \in \mathbb{R} : \langle F, c\rangle = 0$, and that the solution is unique up to an additive constant.
Hint: Use Poincaré's inequality to show the $\widehat{V}$-coercivity of $a(\cdot,\cdot)$.

09 *A Robin problem.* Consider the variational formulation of

$$\begin{aligned} -u''(x) &= f(x) \qquad \text{for } x \in (0,\,1),\\ -u'(0) &= g_0 - \alpha_0\, u(0)\\ u'(1) &= g_1 \end{aligned}$$

with appropriate choices of V_0 and V_g. Show that if $\alpha_0 > 0$, then the corresponding bilinear form is V_0-coercive.
Hint: convince yourself that $\frac{1}{2}\,\|v\|^2_{L^2(0,1)} \leq \|v - v(0)\|^2_{L^2(0,1)} + |v(0)|^2$ and use Friedrichs' inequality to bound the first summand.

In the lecture, the coercivity of the bilinear form

$$a(w,\,v) \;=\; \int_0^1 \big[a(x)\,w'(x)\,v'(x) + b(x)\,w'(x)\,v'(x) + c(x)\,w(x)\,v(x)\,\big]\,dx\,, \tag{2.1}$$

on the space $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ has been shown for the special case $a \equiv 1$, $b \equiv 0$, $c \equiv 0$. In the following three exercises, we consider more general situations. In all situations, you will have to use the estimate

$$a(v,\,v) \;\geq\; a_0\,|v|^2_{H^1(0,1)} + \int_0^1 b(x)\,v'(x)\,v(x)\,dx + c_0\,\|v\|^2_{L_2(0,1)}\,, \tag{2.2}$$

where $a_0 := \inf_{x\in(0,1)} a(x)$ and $c_0 := \inf_{x\in(0,1)} c(x)$.

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10 Show the coercivity of $a(\cdot,\cdot)$ on $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ under the assumptions

$$a_0 > 0\,, \qquad C_F\,\|b\|_{L_\infty(0,1)} < a_0\,, \qquad c_0 \geq 0\,,$$

where C_F is the constant in Friedrichs' inequality.
Hint: Use Cauchy's inequality to show the estimate

$$\int_0^1 b(x)\,v'(x)\,v(x)\,dx \;\geq\; -\|b\|_{L_\infty(0,1)}\,|v|_{H^1(0,1)}\,\|v\|_{L_2(0,1)}$$

and use it in (2.2).

11 Show the coerivity of $a(\cdot, \cdot)$ on the whole space $H^1(0,\, 1)$ under the assumptions

$$a_0 > 0\,, \qquad \|b\|_{L_\infty(0,\,1)} < 2\sqrt{a_0\, c_0}\,, \qquad c_0 > 0\,.$$

Hint: Show that

$$a(v,\, v) \;\geq\; q\big(|v|_{H^1(0,\,1)},\, \|v\|_{L_2(0,\,1)}\big)\,,$$

with

$$q(\xi,\, \eta) \;:=\; a_0\, \xi^2 - \|b\|_{L_\infty(0,\,1)}\, \xi\, \eta + c_0\, \eta^2\,.$$

Finally, show and use that

$$q(\xi,\, \eta) \;\geq\; a_0\, C\, \xi^2\,, \quad \text{and} \quad q(\xi,\, \eta) \;\geq\; c_0\, C\, \eta^2\,,$$

with $C = 1 - \frac{\|b\|^2_{L_\infty(0,\,1)}}{4\, a_0\, c_0}$.

12 Show the coercivity of $a(w,\, v)$ on the space $V_0 = \{v \in H^1(0,\, 1) : v(0) = 0$ under the assumptions

$$a_0 > 0\,, \qquad b(x) = b_0 \geq 0\,, \qquad c_0 \geq 0\,,$$

where b_0 is a constant.
Hint: Show and use that

$$\int_0^1 v'(x)\, v(x)\, dx \;=\; \frac{1}{2}\, v(x)^2\Big|_0^1 \;\geq\; 0 \qquad \forall v \in V_0\,.$$