Robinson's constraint qualifications

Mikalai Zhudro

Institute of Computational Mathematics
Johannes Kepler University
Linz, Austria

December 22, 2009

Supervisor:

Walter Zulehner

Contents

Small Recap

- A set $C \subset X$ is called a convex set if and only if

$$\lambda x + (1-\lambda)y \in C, \forall x, y \in C \text{ and } \lambda \in [0,1].$$

- A multifunction F from a linear space X into a linear space Y is called a *convex* multifunction, if its graph is convex.
- We say F is a *closed convex multifunction* if in addition F is closed.
- *Inverse* of F is a multifunction from Y to X, defined by

$$F^{-1}(y) = \{x \in X : y \in F(x)\}.$$

Metric regularity and r-steepness

Definitions:

- A set-valued map $M : X \rightrightarrows Y$ is said to be *k-regular* ,where $k \in \mathbb{R}$ at $(\overline{x}, \overline{y})$ if for some neighborhood V of$(\overline{x}, \overline{y})$
  $d(x, M^{-1}(y)) \leq kd(y, M(x))$ for $\forall (x, y) \in V$.
- A function $\Phi : Z \longrightarrow \mathbb{R}_+$ is called *subcontinious* if $\forall z_n \longrightarrow z$ satisfying $\Phi(z_n) \longrightarrow 0$ we have $\Phi(z) = 0$
- A function $\Phi : Z \longrightarrow \mathbb{R}_+$ is called *r-step* at $z \in Z$, where $r \in [0, 1)$ if $\forall v \in B(z, \frac{\Phi(x)}{1-r})$ we may find $w \in B(v, \Phi(v))$ such that $\Phi(w) \leq r\Phi(v)$

Metric regularity and r-steepness

Closed-values multifunction (such that M(x) is closed in $X \times Y \ \forall x \in X$) can be characterized by

$$\Phi_M(x, y) = d(y, M(x))$$

Consequence - closed valued mapping M is closed if and only if associated Φ_M is subcontinuous

Proposition. Let $M : X \rightrightarrows Y$ be a closed multifunction and let $(\overline{x}, \overline{y}) \in M$. Assume that $\exists k \geq 0$ and V of $(\overline{x}, \overline{y})$ such that $\forall (x, y) \in V$, $\Phi_y(\cdot) = kd(y, M(\cdot))$ is r-step at x
Then M is $\frac{k}{1-r}$ regular at $(\overline{x}, \overline{y})$ with $d(x, M^-1(y)) \leq \frac{k}{1-r} d(y, M(x))$

Theorem 1.1 Regularity of "equal" functions

Some considerations

- Let's consider multifunction $M : X \rightrightarrows Y$ of type
  $$M_G(x) = \begin{cases} G(x) - C & \text{for } \; x \in A \\ \emptyset & otherwise \end{cases}$$
- $M_F^{-1} = \{x \in X : x \in A \text{ and } G(x) \in y + C\}$, where $A \subset X$,$C \subset Y$ closed convex sets and $G : X \to Y$ a single-valued function
- X,Y - Banach Spaces
- $G : X \to Y$ and $F : X \times U \to Y$ continuous on A and $A \times U$

Theorem 1.1 Regularity of functions

Theorem 1.1 Regularity of functions

Let $(\overline{x}, \overline{y}) \in A \times U$ be : $G(\overline{x}) = F(\overline{x}, \overline{u}) \in C$.
Assume that M_G is k-regular at $(\overline{x}, 0)$, and
$\exists l \in$[0,1/k) and neighborhoods V_0 of $\overline{x}$:
$F(\cdot, u) - G(\cdot)$ is l-lipshitz on $A \cap V_0$ $\forall u \in U$ fixed
Then $\exists k^{'}$ and neighborhood E of $\overline{u}$ such that $M_{F(\cdot,u)}$ is $k^{'}$ regular at $(\overline{x}, 0)$,uniformly for $u \in E$
What is equivalent - for some neighborhood V of $\overline{x}$ and W of $0 \in Y$
$d(x, A \cap F(\cdot, u)^{-1}(y + C)) \leq k' d(F(x, u), y + C)$, for every $x \in A, y \in W$ and $u \in E$

Regularity of linearized function and Robinson constraints

Regularity of convex function

- Following theorem hold
  Let $M : X \rightrightarrows Y$ be a closed convex multifunction and let $(\overline{x}, \overline{y}) \in M$ if $\overline{y} \in core(M(X))$,then M is regular at $(\overline{x}, \overline{y})$

Regularity of linearized function and Robinson constraints

Regularity of linearized function

Letting $G(x) = F(\overline{x}) + D_x F(\overline{x})(x - \overline{x})$ we achieve our main theorem for today's presentation

Let $F : X \to Y$ be strictly differentiable at $\overline{x}$ and continuous on A with $\overline{x} \in A$ and $F(\overline{x}) \in C$.
If $0 \in core[D_x F(\overline{x})(A - \overline{x}) - (C - F(\overline{x}))]$
then M_F is regular at$(\overline{x}, 0)$ and conversely.
What is to say - $\exists k$ positive and for all $x \in A$ sufficiently closed to $\overline{x}$,$d(x, A \cap F^-1(C)) \leq kd(F(x), C)$

Regularity of linearized function and Robinson constraints

Robinson constraints

So far we get Robinson's conditions $0 \in core[DF(\overline{x})(A - \overline{x}) - (C - F(\overline{x}))]$
If we define $F : \mathbb{R}^n \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$, $F = (g_1, ..., g_p, h_1, ..., h_q)$
$C = \mathbb{R}^p \times 0$, $A = \mathbb{R}^n$
we have $A \cap F^-1((y, z) + C) = \{x \in \mathbb{R}^n : g_i(x)_i, h_j(x) = z_j\}$
and Robinson's conditions are Mangasarian-Fromowitz constraint qualification a) $gradh_j(x)_{j=1,...p}$ is linearly dependent
b) there exists $v \in \mathbb{R}^n$ such that
$< \nabla h_j(x), v >= 0$ for all j's
$< \nabla g_i(x), v >< 0$ for all j's

THE END

THANK YOU