Regularity of Convex Multifunction

Esubalewe L. Yedeg

Institute of Computational Mathematics
Johannes Kepler University
Linz, Austria

November 3, 2007

Supervisor:

Univ.-Prof. Dr. Helmut Gfrerer

Contents

1. Introduction
2. Examples
3. Regular value and the open mapping theorem
4. An inversion theorem for convex multifunctions

Introduction

Definitions:

- Let X and Y be two linear spaces. A *multivalued function (multifunction)* from X to Y is a mapping $F : X \rightrightarrows Y$ which assigns a set $F(x) \subset Y$ to each $x \in X$.

Introduction

Definitions:

- Let X and Y be two linear spaces. A *multivalued function (multifunction)* from X to Y is a mapping $F : X \rightrightarrows Y$ which assigns a set $F(x) \subset Y$ to each $x \in X$.
- *Graph* of F
$$\Gamma_F := \{(x, y) \in X \times Y : y \in F(x)\}.$$

Introduction

Definitions:

- Let X and Y be two linear spaces. A *multivalued function (multifunction)* from X to Y is a mapping $F : X \rightrightarrows Y$ which assigns a set $F(x) \subset Y$ to each $x \in X$.
- *Graph* of F
$$\Gamma_F := \{(x, y) \in X \times Y : y \in F(x)\}.$$
- *Effective domain* of F, $\operatorname{dom} F$ is the projection of Γ_F into X.

Introduction

Definitions:

- Let X and Y be two linear spaces. A *multivalued function (multifunction)* from X to Y is a mapping $F : X \rightrightarrows Y$ which assigns a set $F(x) \subset Y$ to each $x \in X$.
- *Graph* of F
$$\Gamma_F := \{(x, y) \in X \times Y : y \in F(x)\}.$$
- *Effective domain* of F, dom F is the projection of Γ_F into X.
- *Range* of F, range F is the projection of Γ_F into Y.

Introduction

Definitions:

- Let X and Y be two linear spaces. A *multivalued function (multifunction)* from X to Y is a mapping $F : X \rightrightarrows Y$ which assigns a set $F(x) \subset Y$ to each $x \in X$.
- *Graph* of F
$$\Gamma_F := \{(x, y) \in X \times Y : y \in F(x)\}.$$
- *Effective domain* of F, dom F is the projection of Γ_F into X.
- *Range* of F, range F is the projection of Γ_F into Y.
- We say F is *closed* if its graph is closed in the product space $X \times Y$.

Introduction

- A set $C \subset X$ is called a convex set if and only if

$$\lambda x + (1-\lambda)y \in C, \forall x, y \in C \text{ and } \lambda \in [0,1].$$

Introduction

- A set $C \subset X$ is called a convex set if and only if

$$\lambda x + (1-\lambda)y \in C, \forall x, y \in C \text{ and } \lambda \in [0,1].$$

- A multifunction F from a linear space X into a linear space Y is called a *convex* multifunction, if its graph is convex.

Introduction

- A set $C \subset X$ is called a convex set if and only if

$$\lambda x + (1-\lambda)y \in C, \forall x, y \in C \text{ and } \lambda \in [0,1].$$

- A multifunction F from a linear space X into a linear space Y is called a *convex* multifunction, if its graph is convex.
- We say F is a *closed convex multifunction* if in addition F is closed.

Introduction

- A set $C \subset X$ is called a convex set if and only if

$$\lambda x + (1-\lambda)y \in C, \forall x, y \in C \text{ and } \lambda \in [0,1].$$

- A multifunction F from a linear space X into a linear space Y is called a *convex* multifunction, if its graph is convex.
- We say F is a *closed convex multifunction* if in addition F is closed.
- *Inverse* of F is a multifunction from Y to X, defined by

$$F^{-1}(y) = \{x \in X : y \in F(x)\}.$$

Introduction

Theorem 1 *The following statements are equivalent*

i. $F : X \rightrightarrows Y$ *is a convex multifunction.*

ii. *the inclusion*

$$F(\lambda x_1 + (1-\lambda)x_2) \supset \lambda F(x_1) + (1-\lambda)F(x_2)$$

holds $\forall x_1,\ x_2 \in X, \lambda \in [0,1]$.

Examples

In finite dimensional space: Let $X = \mathbb{R}^n$

1. If $f : X \to \mathbb{R}$ is a convex function and $\mathbb{R}_+$ is the set of nonnegative real numbers, then the mapping F defined by

$$F(x) := f(x) + \mathbb{R}_+, \quad \text{is a convex multifunction}$$

Examples

In finite dimensional space: Let $X = \mathbb{R}^n$

1. If $f : X \longrightarrow \mathbb{R}$ is a convex function and $\mathbb{R}_+$ is the set of nonnegative real numbers, then the mapping F defined by

$$F(x) := f(x) + \mathbb{R}_+, \quad \text{is a convex multifunction}$$

2. Let A be an $m \times n$ and $b \in \mathbb{R}^m$. The mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$f(x) = Ax - b$$

is a convex multifunction.

Examples

In finite dimensional space: Let $X = \mathbb{R}^n$

1. If $f : X \longrightarrow \mathbb{R}$ is a convex function and $\mathbb{R}_+$ is the set of nonnegative real numbers, then the mapping F defined by

$$F(x) := f(x) + \mathbb{R}_+, \quad \text{is a convex multifunction}$$

2. Let A be an $m \times n$ and $b \in \mathbb{R}^m$. The mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$f(x) = Ax - b$$

is a convex multifunction.

3. Let $g_i : X \longrightarrow \mathbb{R}, \quad i = 1, \dots, m$; be convex functions, then

$$F(x) = \begin{pmatrix} Ax - b \\ g(x) \end{pmatrix} + \begin{pmatrix} 0_m \\ \mathbb{R}^m_+ \end{pmatrix}, \quad g = (g_1, \dots, g_m)^T$$

is a convex multifunction.

Examples

In infinite dimensional spaces:

1. Let X and Y be arbitrary linear spaces, $P : X \to Y$ be a linear operator and $y_0 \in Y$, then $F : X \rightrightarrows Y$ defined by

$$F(x) = Px - y_0$$

   is a convex multifunction.

Examples

In infinite dimensional spaces:

1. Let X and Y be arbitrary linear spaces, $P : X \to Y$ be a linear operator and $y_0 \in Y$, then $F : X \rightrightarrows Y$ defined by

$$F(x) = Px - y_0$$

   is a convex multifunction.

2. Let $\Omega \subset \mathbb{R}^d$ and $X \subset L^2(\Omega)$, for fixed $\phi \in \text{Ł}^2(\Omega)$ define

$$C = \{x \in X : x(t) \leq \phi(t) \ \ \forall t \in \Omega\}$$

   then the mapping $F : X \rightrightarrows L^2(\Omega)$ defined by

$$F(x) := x - C$$

   is a convex multifunction.

Examples

3. Let P and C be as defined in the previous examples. The mapping $F : X \rightrightarrows Y$ defined by

$$F(x) = \begin{pmatrix} Px - y_0 \\ x \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ C \end{pmatrix}$$

is a convex multifunction.

Examples

3. Let P and C be as defined in the previous examples. The mapping $F : X \rightrightarrows Y$ defined by

$$F(x) = \begin{pmatrix} Px - y_0 \\ x \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ C \end{pmatrix}$$

is a convex multifunction.

4. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function, $G : X \longrightarrow L^2(\Omega)$ defined by $G(x)(t) := g(x(t))$ and

$$\bar{C} = \{x \in L^2(\Omega) : x(t) \leq \phi(t) \ \ \forall t \in \Omega\}, \ \ \phi \in L^2(\Omega)$$

then $F : X \rightrightarrows L^2(\Omega)$ defined by $F(x) = G(x) - \bar{C}$ is a convex multifunction.

Regular Values and the Open Mapping Theorem

Lemma 2 *Let X and Y be normed linear spaces, and let C be a closed convex set in $X \times Y$. Denote by P_X and P_Y the projections from $X \times Y$ into X and Y respectively, and suppose that $P_X(C)$ is bounded.*

a) If X is complete, then $\text{int}\, \text{cl} P_Y(C) = \text{int}\, P_Y(C)$.

b) If X is reflexive, then $P_Y(C)$ is closed.

Regular Values and the Open Mapping Theorem

Definition Let F be a convex multifunction from a linear space X into a linear space Y, and let $y \in \text{range } F$. We say that y is a *regular value* of F if y is an internal point of range F, and a singular value of F otherwise.

Regular Values and the Open Mapping Theorem

Definition Let F be a convex multifunction from a linear space X into a linear space Y, and let $y \in$ range F. We say that y is a *regular value* of F if y is an internal point of range F, and a singular value of F otherwise.

Theorem 3 *Let X and Y be Banach spaces, and let F be a closed convex multifunction from X into Y. Let y be an internal point of range F. Then for each $x \in F^{-1}(y)$ there exists a positive η such that for each $\lambda \in [0, 1]$*

$$F(x + \lambda B_X) \supset y + \lambda \eta B_Y.$$

An Inversion Theorem for Convex Multifunction

Lemma 4 *Let X and Y be Banach spaces, with a closed convex multifunction F from X into Y. Suppose that $x_0 \in X$ and $y_0 \in Y$ are such that for some bounded convex set $D \subset Y$ and some real numbers η and δ with* $0 \leq \delta \leq \eta$,

$$y_0 + \eta D \subset F(x_0 + B_X) + \delta D.$$

Then the following inclusions hold:

a) If X is complete, then

$$y_0 + int(\eta - \delta)D \subset intF(x_0 + B_X),$$

b) If X is reflexive, then

$$y_0 + (\eta - \delta)D \subset F(x_0 + B_X).$$

An Inversion Theorem for Convex Multifunction

Convention:

for $x \in X$ and $A \subset X, d[x, A] := \inf\{||x - a|| : a \in A\}(+\infty \text{ if } A = \emptyset)$

Theorem 5 *Let X be Banach spaces and Y be a normed space, and let F be a closed convex multifunction from X into Y. Let $y_0 \in F(x_0)$, and suppose that for some $\eta > 0$ and some $\delta \in [0, \eta)$,*

$$y_0 + \eta B_Y \subset F(x_0 + B_X) + \delta B_Y.$$

Then for any $x \in X$ and any $y \in y_0 + int)\eta - \delta)B_Y$,

$$d[x, F^{-1}(y)] \leq (\eta - \delta - ||y - y_0||)^{-1}(1 + ||x - x_0||)d[y, F(x)].$$

THE END

THANK YOU