Openness and Metric Regularity of a Multifunction

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Contents

Introduction

Definitions:

- Let X and Y be two linear spaces. A *multivalued function (multifunction)* $\psi : X \to 2^Y$ is a mapping from X into the set 2^Y of subsets of Y
- *Graph* of ψ
$$gph(\psi) := \{(x, y) \in X \times Y : y \in \psi(x), x \in X\}.$$
- *Effective domain* of ψ, dom ψ is the projection of $gph(\psi)$ into X.
- *Range* of ψ, range ψ is the projection of $gph(\psi)$ into Y.
- We say ψ is *closed* if its graph is closed in the product space $X \times Y$.

Introduction

- A set $C \subset X$ is called a convex set if and only if

$$\lambda x + (1 - \lambda) y \in C, \forall x, y \in C \text{ and } \lambda \in [0, 1].$$

- A multifunction ψ from a linear space X into a linear space Y is called a *convex* multifunction, if its graph is convex.
- We say ψ is a *closed convex multifunction* if in addition ψ is closed.
- *Inverse* of ψ is a multifunction from Y to X, defined by

$$\psi^{-1}(y) := \{x \in X : y \in \psi(x)\}.$$

Introduction

Recall:

Lemma 1 *Let X and Y be normed linear spaces, and let C be a closed convex set in $X \times Y$. Denote by P_X and P_Y the projections from $X \times Y$ into X and Y respectively, and suppose that $P_X(C)$ is bounded.*

a) If X is complete, then $int\, cl P_Y(C) = int\ P_Y(C)$.

b) If X is reflexive, then $P_Y(C)$ is closed.

Introduction

Recall:

Theorem 2 ***(Generalized Open mapping theorem)***
Let X and Y be two linear spaces, and let $\psi : X \to 2^Y$ be a closed convex multifunction. Let $y \in int(range\psi)$. Then $y \in int\psi(B_X(x,r))$ for every $x \in \psi^{-1}(y)$ and $r > o$

Introduction

Recall:

Theorem 3 *The following statements are equivalent*

i. $\psi : X \to 2^Y$ *is a convex multifunction.*

ii. *the inclusion*

$$\psi(tx_1 + (1-t)x_2) \supset t\psi(x_1) + (1-t)\psi(x_2)$$

holds $\forall x_1,\ x_2 \in X, t \in [0,1]$.

Examples

In finite dimensional space: Let $X = \mathbb{R}^n$

1. If $f : X \to \mathbb{R}$ is a convex function and $\mathbb{R}_+$ is the set of nonnegative real numbers, and the mapping ψ defined by

$$\psi(x) := f(x) + \mathbb{R}_+,$$

$$0 \in \psi(x) := f(x) + \mathbb{R}_+,$$

2. Let A be an $m \times n$ and $b \in \mathbb{R}^m$. The mapping $\psi : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\psi(x) := \{Ax - b\}$$

$$0 \in \psi(x) := \{Ax - b\}$$

The problem

The problem: Find $x \in X$ such that

1. $0 \in \psi(x) := f(x) + \mathbb{R}_+$ for a convex function $f : X \to \mathbb{R}$
2. $0 \in \psi(x) := \{Ax - b\}$ where A is $m \times n$ and $b \in \mathbb{R}^m$.
3. combining (1) and (2)

To show: the connection

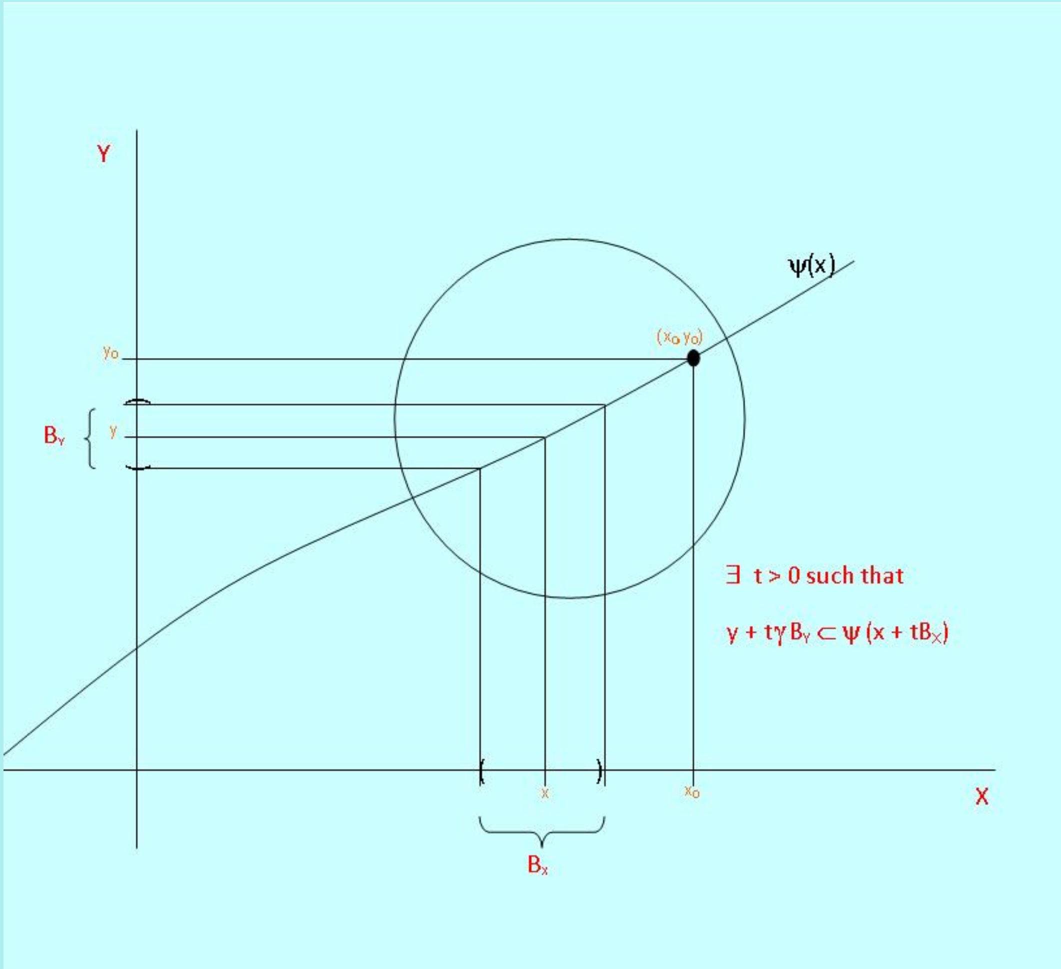
$$0 \in int(range(\psi)) \Leftrightarrow y + t\gamma B_Y \subset \psi(x + tB_X) \Leftrightarrow dist(x, \psi^{-1}(y)) \leq c\, dist(y, \psi(x))$$

Openness of a Multifunction

Definition 1: We say that the multifunction is open at a point $(x_0, y_0) \in gph(\psi)$, at a (linear) rate $\gamma > 0$, if there exist $t_{max} > 0$ and a neighbourhood N of (x_0, y_0) such that for all $(x, y) \in gph(\psi) \cap N$ and all $t \in [0, t_{max}]$ the following inclusion holds:

$$y + t\gamma B_Y \subset \psi(x + tB_X) \tag{1}$$

Openness of a Multifunction



Openness of a Multifunction

Proposition 1: If the multifunction ψ is convex, then ψ is open at a point $(x_0, y_0) \in gph(\psi)$ if and only if there exist positive constant η and ν such that

$$y_0 + \eta B_Y \subset \psi(x_0 + \nu B_X) \tag{2}$$

Proof:

strategy:

i) If ψ is a convex function,
$\psi(tx_1 + (1-t)x_2) \supset t\psi(x_1) + (1-t)\psi(x_2)$ (from first presentation)

ii) $y_0 + \eta B_Y \subset \psi(x_0 + \nu B_X), for\ \eta, \nu > 0$

" $\Leftarrow$ "

Clearly (2) follows from (1) by taking $\nu = t_{max}$ and $\eta = \gamma t_{max}$

" $\Rightarrow$ "

suppose ψ is convex,

w.l.o.g we can assume that $x_0 = 0$ and $y_0 = 0$

Let $(x, y) \in gph(\psi) \cap N$ for $N := \nu B_X \times \frac{1}{2}\eta B_Y \qquad (*)$

using $y \in \psi(x)$ and considering a ball in Y with center y and radius $\frac{1}{2}t\eta$

we have for any $t \in [0, 1]$ that

$$\begin{aligned} y + \frac{1}{2}t\eta B_Y &= (1-t)y + t(y + \frac{1}{2}t\eta B_Y) \\ &\subset (1-t)y + t\eta B_Y \\ &\subset (1-t)\psi(x) + t\psi(\nu B_X)\ \{from(ii)\} \\ &\subset \psi((1-t)x + t\nu B_X)\ \{from(i)\} \\ &\subset \psi(x + 2t\nu B_X) \end{aligned} \tag{3}$$

This implies (1), with N defined by $(*)$ and $\gamma = \frac{\eta}{4\nu}, t_{max} = 2\nu$

Openness of a Multifunction

Proposition 2: Suppose that the multifunction $\psi : X \longrightarrow 2^Y$ is closed and convex. Then ψ is open at (x_0, y_0) if and only if $y_0 \in int(range\psi)$

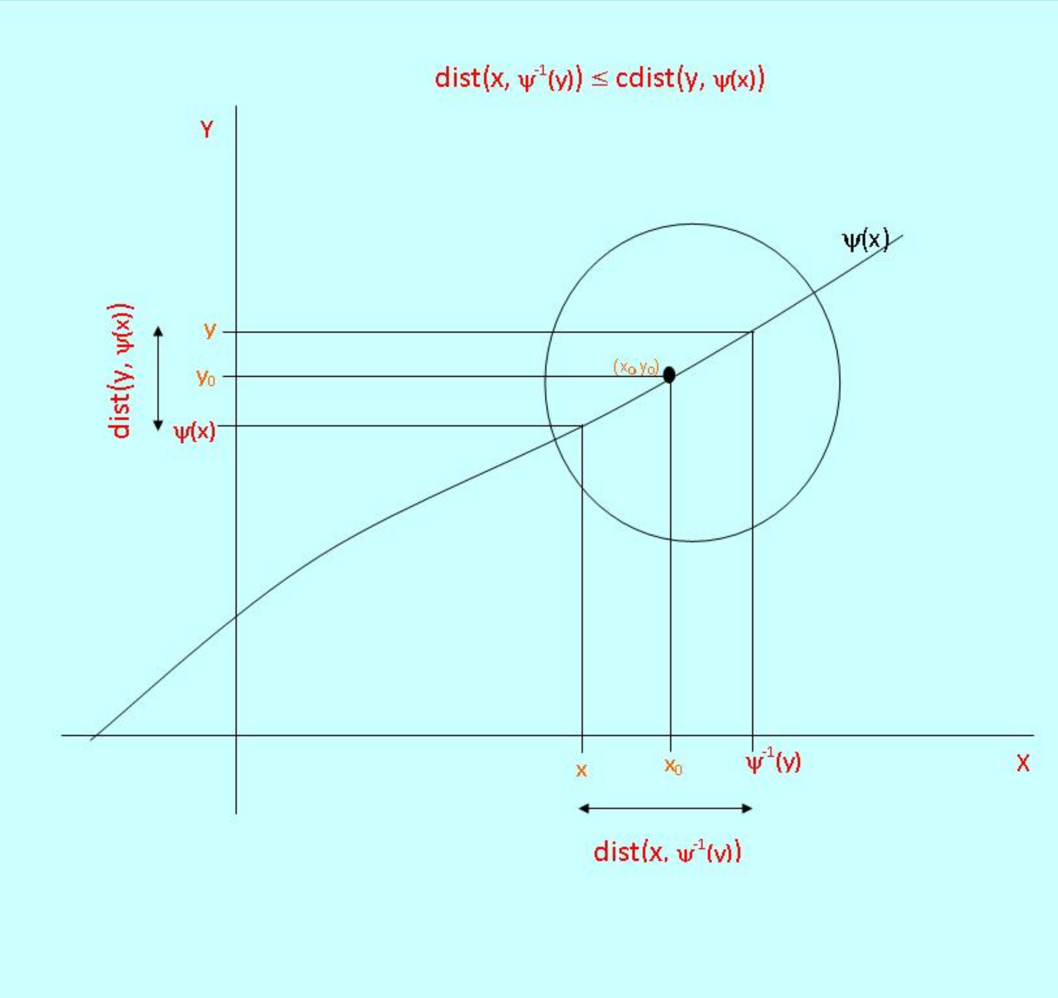
proof: This is just a consequence of the *Generalized open mapping theorem*

Metric Regularity

Definition 2: We say that the multifunction $\psi : X \longrightarrow 2^Y$ is metric regular at a point $(x_0, y_0) \in gph(\psi)$ at a rate c, if for all (x, y) in a neighbourhood of (x_0, y_0)

$$dist(x, \psi^{-1}(y)) \leq c \; dist(y, \psi(x)) \tag{4}$$

Metric Regularity



Metric Regularity

Theorem 4 *The multifunction* $\psi : X \rightarrow 2^Y$ *is metric regular at a point* $(x_0, y_0) \in gph(\psi)$*, at a rate* c*, if and only if* ψ *is open at* (x_0, y_0) *at the rate* $\gamma := c^{-1}$.

proof:
$" \Rightarrow "$
suppose ψ is open at $(x_0, y_0), \gamma > 0, t_{max} > 0, N$ according to definition (1)
w.l.o.g, we can assume that $N = \epsilon_x B_x \times \epsilon_x B_y$
reducing t_{max} if necessary, we can also assume that

$$t_{max}\gamma \leq \frac{1}{2}\epsilon'_y \qquad \qquad . \ . \ . \ . \ . \ (i)$$

Metric Regularity

Let (x, y) be such that $||x - x_0|| < \epsilon'_x, ||y - y_0||| < \epsilon'_y \ . \ . \ . \ . \ . \ (ii)$
where $\epsilon'_x, \epsilon'_y > 0$ satisfying $\epsilon'_x \leq \epsilon_x$ and $\gamma\epsilon'_x + \epsilon'_y \leq t_{max}\gamma. \ . \ . \ . \ (iii)$
$\Rightarrow \epsilon'_y \leq \frac{1}{2}\epsilon_y$
we can claim that definition of metric regularity holds with $c = \gamma^{-1}$

indeed since $\epsilon'_y \leq \epsilon_y$ and ψ is open at $(x_0, y_0), \exists\ x^* \in \psi^{-1}(y)$ such that $||x^* - x_0|| \leq \gamma^{-1}||y - y_0||$

it follows that

$$\begin{aligned} dist\ (x, \psi^{-1}(y)) &\leq ||x - x^*|| \leq ||x - x_0|| + \gamma^{-1}||y - y_0|| \\ &\leq \epsilon'_x + \gamma^{-1}\epsilon'_y \end{aligned} \tag{5}$$

Metric Regularity

Consequently, if

$$dist(y, \psi^{-1}(x)) \geq \gamma(\epsilon_x' + \gamma^{-1}\epsilon_y') = \gamma\epsilon_x' + \epsilon_y' \text{ in particular if } \psi(x) = \phi$$

then our claim holds.
Otherwise in view of equation (iii), for small $\alpha > 0 \exists y_\alpha \in \psi(x)$ such that

$$||y - y_\alpha|| \leq dist(y, \psi(x)) + \alpha < \gamma\epsilon_x' + \epsilon_y' \leq t_{max}\gamma$$

Then due to $eqn(i), (ii), (iii)$, we have

$$||y_\alpha - y_0|| \leq ||y_\alpha - y|| + ||y - y_0|| < t_{max}\gamma + \epsilon_y' \leq \epsilon_y. \ . \ . \ . \ . \ . \ \text{(iv)}$$

$\therefore (x, y_\alpha) \in gph(\psi) \cap N. \ . \ . \ . \ . \ . \ . \ . \ . \ .$ (v)

Metric Regularity

Thus combining (iv) and openness of ψ at (x_0, y_0)
$\Rightarrow \exists\, x' \in \psi^{-1}(y)$ such that $||x' - x|| \leq \gamma^{-1}||y - y_\alpha||$
it follows that

$$dist(x, \psi^{-1}(y)) \leq ||x' - x|| \leq \gamma^{-1}||y - y_\alpha|| \leq \gamma^{-1}dist(y, \psi(x)) + \gamma^{-1}\alpha$$

Since $\alpha > 0$ is arbitrary, definition (2) follows with $c = \gamma^{-1}$

" $\Leftarrow$ "
Suppose ψ is metric regular at (x_0, y_0) at rate $c > 0$
Let $(x, y) \in gph(\psi)$, $z \in Y$ such that $||y - z|| < t\ c^{-1}$

Metric Regularity

Then for (x, y) sufficiently close to (x_0, y_0) and $t > 0$ small enough we have

$$dist(x, \psi^{-1}(z)) \leq c\ (dist(z, \psi(x))) \leq c\ ||z - y|| < t$$

$\Rightarrow \exists\ w \in \psi^{-1}(z)$ such that $||w - x|| < t$
hence $z \in \psi(x + t\ B_x)$ $\quad \square$

Conclusion - The consequence

1. Given that $y_0 \in intrange(\psi)$,
   Let $y_0 = 0$ Let A be an $m \times n$ and $b \in \mathbb{R}^m$.
   $0 \in \psi(x) := \{Ax - b\}(\Leftrightarrow Ax = b)$, then
   $range(\psi) = \{Ax - b : x \in \mathbb{R}^n\} \subset \mathbb{R}^m$

   – if $rank(A) < m$ then 0 is not a regular value

   – $rank(A) = m$ then $\quad 0 \in intrange(\psi)$ is a regular value,

   hence suppose $x_0 \in X : (x_0, y_0) \in gph(\psi)$ and $\forall(x, y)$ in the nbhd of (x_0, y_0)
   Let $Ax_0 - b = y_0$ and $A\bar{x} - b = \bar{y}$ such that for sufficiently small ϵ we have $|x - x_0| \leq \epsilon$ and $|y - y_0| \leq \epsilon$ then

$$|x - \bar{x}| \leq c|y - \bar{y}| \qquad \Rightarrow \text{Metric regularity}$$

Conclusion - The consequence

2. Given $y_0 \in intrange(\psi)$
   Let $y_0 = 0$
   $0 \in \psi(x) := g(x) + \mathbb{R}_+, g : X \longrightarrow \mathbb{R}$ convex $(\Leftrightarrow \exists x_0 : g(x_0) \leq 0)$
   Suppose $(x_0, y_0) \in \psi(x)$
   $Inf|x - x_0| \leq c\; inf|g(x) + \mathbb{R}_+|$ where $g(x) + \mathbb{R}_+ \in [g(x), \infty)$

$$\begin{aligned} \text{but} \;\; c\; inf|g(x) + \mathbb{R}_+| &\Leftrightarrow inf|z| \; s.t \; z \in g(x) + \mathbb{R}_+ \\ &\Rightarrow inf\{|z| : z \in [g(x), \infty)\} \\ &= \begin{cases} g(x) & \text{for } g(x) \geq 0 \\ 0 & \text{for } g(x) < 0 \end{cases} \\ &= \;\; max(0, g(x)) \end{aligned} \tag{6}$$

$\therefore inf_{g(x_0) \leq 0}|x - x_0| \leq c\; max(0, g(x))$

Conclusion - The consequence

Theorem 5 *(Robinson- Ursescu Stability Theorem) Let* $\psi : X \to 2^Y$ *be a closed convex multifunction. Then* ψ *is metric regular at* $(x_0, y_0) \in gph(\psi)$ *if and only if the regularity condition* $y_0 \in int(range\psi)$ *holds.*

More precisely, suppose that (2)*(proposition 2) is satisfied, and let* (x, y) *be such that*

$$||x - x_0|| < \frac{1}{2}\nu,\ ||y - y_0|| < \frac{1}{8}$$

then (3)*(definition of Metric Regularity) holds with constant* $c = \frac{4\nu}{\eta}$

Conclusion - The consequence

proof:

" $\Rightarrow$ " The equivalence between metric regularity and the regularity condition $y_0 \in int(range\psi)$ is just a consequence of what we have done in proposition 2 and theorem 4

" $\Leftarrow$ " we only need to check the estimates of the constants

THE END

THANK YOU