Generalized Penalty Methods

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Outline

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Optimization problem

$$(P) \qquad \min_{z \in Z} J(z)$$

subject to

$$Ez = 0$$

$$g_i(z)(x) \leq \varphi_i(x) \qquad \text{for a.e. } x \in \Omega_i,\ i = 1, \dots m$$

- Z is a Hilbert space
- $E : Z \to V$ bounded and linear

$$Z_E := \{z \in Z \ : \ Ez = 0\}$$

- $g_i : Z \to L^{r_i}(\Omega_i)$ and $\varphi_i \in L^{r_i}(\Omega_i)$

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Model problem

$$\text{(MP)} \qquad \min_{z=(y,u)} J(z) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)}$$

subject to

$$\begin{aligned} -\Delta y(x) &= u(x) && \text{in } \Omega \\ y(x) &= 0 && \text{on } \partial\Omega \\ u(x) &\leq \varphi_u(x) && \text{f.a.e } x \in \Omega \\ y(x) &\leq \varphi_y(x) && \text{f.a.e } x \in \Omega \\ |\nabla y(x)|_2 &\leq \varphi_g(x) && \text{f.a.e } x \in \Omega \end{aligned}$$

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Assumptions on the Problem

(P1) There is a feasible point $z_f \in Z_F$

(P2) The cost functional J is **convex** and lower semi continuous (**l.s.c.**) on Z_E

(P3) $\exists \alpha > 0$ s.t. $\forall z_1, z_2 \in Z_E$:

$$J(z_2) \geq J(z_1) + J'(z_1, z_2 - z_1) + \frac{\alpha}{2}\|z_2 - z_1\|^2$$

(P4) For every closed convex set $U \subset L^{r_i}(\Omega_i)$ the pre-image $\{z \in Z_E \;:\; g_i(z) \in U\}$ is **closed**. Further all g_i are convex in z (for a.e. $x \in \Omega_i$)

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Outline

Trivial barrier function

(P) is equivalent to

$$\min_{z \in Z_E} J(z) + \Psi(z) := J(z) + \sum_{i=1}^{m} \mathcal{I}_{(-\infty,0]}(g_i(z)(x) - \varphi_i(x))\mathrm{d}x,$$

where the indicator function $\mathcal{I}_{(-\infty,0]}$ is given by

$$\mathcal{I}_{(-\infty,0]}(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ \infty & \text{if } t > 0. \end{cases}$$

Generalized penalty function

$$(P_q) \qquad \min_{z \in Z_E} J_q(z) := J(z) + \Psi_q(z) := J(z) + \sum_{i=1}^{m} \psi_{i,q_i}(g_i(z)(x) - \varphi_i(x))\mathrm{d}x$$

Consider $q \in Q$ and $q \to \overline{q}$

Idea: $\psi_{i,q_i} \to \mathcal{I}_{(-\infty,0]}$ for $q \to \overline{q}$

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Convergence result

Theorem (Gfrerer (2.1): Convergence)

Assume (P1) – (P4) and that the ***penalty function*** Ψ_q *fulfills for some* $q \in Q\backslash\{\overline{q}\}$*:*

(1) $\Psi_q : Z \to \mathbb{R} \cup \{\infty\}$ *is well defined and convex and l.s.c on* Z_E

(2) $\Psi_q(z_f) < \infty$ *(finite at some feasible point)*

(3) $\Psi_q(z) \geq \sum_{i=1}^m (b_{i,q}\langle \eta_i^*, z\rangle + a_{i,q})$ *for all* $z \in Z_E$

Then:

- (P_q) *has a unique solution (called* $\overline{z}_q$*)*

Convergence result (cont.)

Theorem (Gfrerer (2.1): Convergence)

*Assume (P1) – (P4) and that the **penalty function** fulfills:*

(1) – (3) *as before for all $q \in Q\backslash\{\overline{q}\}$ uniformly*

(η_i^* does not depend on q; $a_{i,q}$, $b_{i,q} \to 0$ for $q \to \overline{q}$)

(4) *for every $\hat{z} \in Z_E\backslash Z_F$ (**not feasible** for (P)):*
*$\forall R > 0$ there exist **neighborhoods** $U_{\hat{z}} \subset Z_E$ and $U_{\overline{q}} \subset Q$*

$$\inf_{q \in U_{\overline{q}}} \inf_{z \in U_{\hat{z}}} \Psi_q(z) \geq R$$

(5) *for every $\hat{z} \in Z_F$ (**feasible** for (P)):*
there is some family $(\hat{z}_q)_{q \in Q\backslash\{\overline{q}\}} \subset Z_E$ with $\lim_{q \to \overline{q}} \hat{z}_q = \hat{z}$
such that $\limsup_{q \to \overline{q}} \Psi_q(\hat{z}_q) \leq 0$

Then:

- $\lim_{q \to \overline{q}} \overline{z}_q = \overline{z}$ *(convergence to the exact solution of (P))*

Convergence result (cont.)

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Construction of generalized penalty functions

- Know properties that should be fulfilled by Ψ_q
- Goal: construct functions $\psi_{i,q}$ such that these properties are fulfilled

Some examples for generalized penalty functions

- Quadratic penalty function:

$$\psi_\kappa(t) = \kappa \max\{0, t\}^2$$

- Logarithmic barrier function:

$$\psi_\kappa(t) = \begin{cases} -\kappa \ln(-t) & \text{if } t < 0 \\ \infty & \text{if } t \geq 0 \end{cases}$$

- Combined logarithmic-quadratic penalty function:

$$\psi_{\kappa,\epsilon}(t) = \begin{cases} -\kappa \ln(-t) & \text{if } t \leq -\epsilon \\ \kappa(-\ln(\epsilon) + \frac{t+\epsilon}{\epsilon} + \frac{(t+\epsilon)^2}{2\epsilon^2}) & \text{if } t > -\epsilon \end{cases}$$

where $Q := \{(\kappa, \epsilon) \in \mathbb{R}_+^\infty \ : \ \kappa \geq a\epsilon^{3/2} \text{ and } \kappa^{1/2} \ln \epsilon > -b\}$ $(a, b > 0 \text{ const})$

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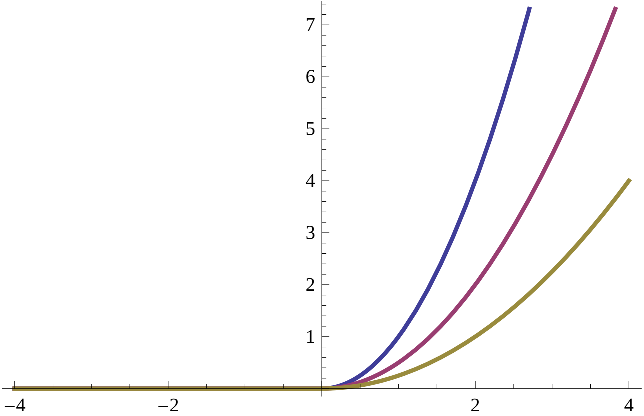

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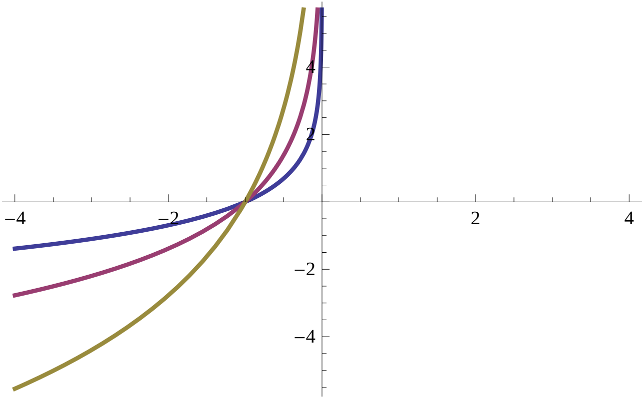

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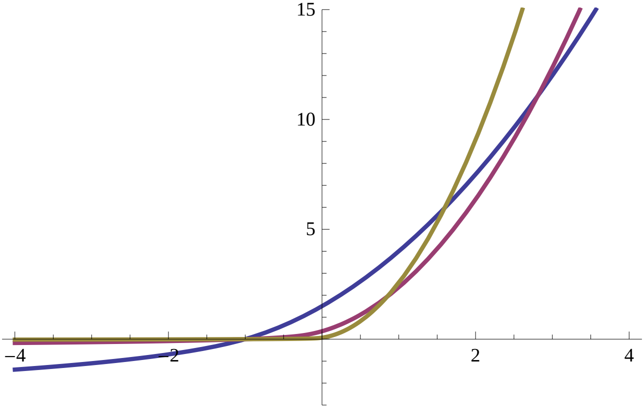

Basic assumptions on generalized penalty functions

(Q1) $\psi_{i,q}$ is **convex, l.s.c. and increasing** and

$$(-\infty,0) \subset \mathrm{dom}\psi_{i,q}$$

(Q2) $\lim_{q\to\overline{q}} \psi_{i,q}(t) = \mathcal{I}_{(-\infty,0]}(t)$ pointwise for all $t \neq 0$

(PQ1) One of the following conditions hold:

- $g_i(z_f) \le \varphi_i - \delta$ for $\delta > 0$ (in the **interior**)
- $0 \in \mathrm{dom}\psi_{i,q}$ for all $q \in Q\backslash\{\overline{q}\}$ and $\lim_{q\to\overline{q}} \psi_{i,q}(0) = 0$ (*good* behavior for 0)

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We notice that (Q1), (Q2) and (PQ1) are fulfilled if

- $\psi_{i,q}$ is **convex, l.s.c. and increasing**,

$$(-\infty, 0] \subset \text{dom}\psi_{i,q}$$

- $\lim_{q\to\overline{q}} \psi_{i,q}(t) = \mathcal{I}_{(-\infty,0]}(t)$ pointwise for all t

These conditions hold for the **quadratic** penalty function and the **combined** penalty function.

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We notice that (Q1), (Q2) and (PQ1) are fulfilled for the **logarithmic barrier function** iff **there exists an interior point.**

Convergence result

Corollary

Assume that (P1)–(P4), (Q1), (Q2) and (PQ1) hold. Then for each $q \in Q\backslash\{\overline{q}\}$ the problems (P_q) each have a unique solution $\overline{z}_q$ and

$$\lim_{q \to \overline{q}} \overline{z}_q = \overline{z},$$

i.e., the solutions converge to the solution of (P).

One can show that the combination of (Q1), (Q2) and (PQ1) implies the assumptions of the main theorem.

Outline

A-priori error estimates

Theorem (Gfrerer (3.3): Error estimate)

Assume *(P1) – (P4), (Q1), (Q2), (PQ1) and*

(Q3') *For each* $q \in Q \backslash \{\overline{q}\}$ *there are* $c_{i,q} \geq 0$ *and* $d_{i,q} \geq 0$ *s.t.*

$$\psi_{i,q}(t/2) \leq c_{i,q}\psi_{i,q}(t) + d_{i,q} \qquad \forall t \in [-1, 0)$$

A-priori error estimates

Theorem (Gfrerer (3.3): Error estimate)

Then: *there is a neighborhood $U_{\overline{q}} \subset Q$ and a constant L such that for all $q \in U_{\overline{q}}$ we have*

$$\|\overline{z}_q - \overline{z}\|_Z^2 \leq \frac{2}{\alpha}\left(L\,\mathrm{dist}(\overline{z}_q, Z_F) + \sum_{i=1}^{m} \int_{\Omega_i} \psi_{i,q}^{\#}(g_i(\overline{z}_q)(x) - \varphi_i(x), -(g_i(\overline{z}_q)(x) - \varphi_i(x)))\mathrm{d}x \right),$$

where

$$\psi_{i,q}^{\#}(u,h) := \begin{cases} \psi_{i,q}'(u,h) & \text{if } u \in \mathrm{int\ dom}\ \psi_{i,q} \\ \lim_{t\downarrow 0} \frac{\psi_{i,q}(u+th)-\psi_{i,q}(u)}{t} \in \mathbb{R} \cup \{-\infty\} & \text{if } u \in \mathrm{bd\ dom}\ \psi_{i,q}, h < 0 \\ 0 & \text{otherwise} \end{cases}$$

Application to combined penalty function

- Combined logarithmic-quadratic penalty function:

$$\psi_{\kappa,\epsilon}(t) = \begin{cases} -\kappa \ln(-t) & \text{if } t \leq -\epsilon \\ \kappa(-\ln(\epsilon) + \frac{t+\epsilon}{\epsilon} + \frac{(t+\epsilon)^2}{2\epsilon^2}) & \text{if } 0 > t > -\epsilon \end{cases}$$

- We obtain the following derivatives:

$$\psi^{\#}_{\kappa,\epsilon}(t,-t) := \begin{cases} \kappa & \text{if } t \leq -\epsilon \\ -\kappa(\frac{2t}{\epsilon} + \frac{t^2}{\epsilon^2}) \leq \kappa & \text{if } t > -\epsilon \end{cases}$$

- Plugging into the last theorem shows:

$$\|\overline{z}_{(\kappa,\epsilon)} - \overline{z}\|_Z^2 \leq \frac{2}{\alpha}\left(L \operatorname{dist}(\overline{z}_{(\kappa,\epsilon)}, Z_F) + \sum_{i=1}^{m} \kappa \, |\Omega_i| \right)$$

Application to combined penalty function

- Combined logarithmic-quadratic penalty function:

$$\psi_{\kappa,\epsilon}(t) = \begin{cases} -\kappa \ln(-t) & \text{if } t \le -\epsilon \\ \kappa(-\ln(\epsilon) + \frac{t+\epsilon}{\epsilon} + \frac{(t+\epsilon)^2}{2\epsilon^2}) & \text{if } 0 > t > -\epsilon \end{cases}$$

- We obtain the following derivatives:

$$\psi^{\#}_{\kappa,\epsilon}(t,-t) := \begin{cases} \kappa & \text{if } t \le -\epsilon \\ -\kappa(\frac{2t}{\epsilon} + \frac{t^2}{\epsilon^2}) \le \kappa & \text{if } t > -\epsilon \end{cases}$$

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Summary

- For all three cases some error estimates are possible
- The combined logarithmic-quadratic penalty function can be bounded by essential the same bounds as the logarithmic barrier functions but it works without having an interior point
- For the combined logarithmic-quadratic penalty function κ and ϵ can be adjusted to ensure both contributions of the error to be small

Outline

Lagrange Multipliers

- Sometimes Lagrange multipliers are of our interest (e.g. Stokes problem: multiplier p is the pressure)
- Methods based on the optimality system deliver the solution and the Lagrange multipliers
- For penalty methods the existence of Lagrange multipliers is not required and the multipliers are not computed during the iteration
- If we assume a **constraint qualification condition** ensuring the **existence** of multipliers, the multipliers **can be approximated** using the generalized penalty methods

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Additional assumption

(Q3) For each $q \in Q \backslash \{\overline{q}\}$ we assume:

- $\psi_{i,q}$ to be differentiable on $\mathbb{R}$
- $\limsup_{t \to \infty} t^{-r_i} \psi_{i,q}(t) < \infty$
- $\lim_{q \to \overline{q}} \psi_{i,q}(0) = 0$

Notice:

- (Q3) is fulfilled by the **quadratic** penalty method, by the **combined quadratic-logarithmic** penalty method but not by the **logarithmic barrier** method
- (Q3) implies the (PQ1) and (Q3')
- (Q3) and (Q1) imply $|\psi_{i,q}(t)| \leq C_{i,q}(1 + |t|^{r_i})$ for all t. This implies $\int_{\Omega_i} \psi_{i,q}(u(x))dx$ to be l.s.c., convex, real-valued on $L^{r_i}(\Omega_i)$ and **differentiable**

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Linearize the problem

Therefore we can linearize the problem (P_q).
We know that $0 \leq J_q'(\overline{z}_q, z - \overline{z}_q)$ holds.
So we can derive that $\overline{z}_q$ is the unique solution of

$$\min_{z \in Z_E} J(z) + \langle y_q^*, G_i(z) \rangle,$$

where

- $\langle y_q^*, y \rangle := \sum_{i=1}^m \langle l_{i,q}^*, y_i \rangle$
- $G(z) := (g_1(z), \dots, g_n(z))$
- $l_{i,q}^* \in L^{r_i}(\Omega_i)^*$ such that

$$\langle l_{i,q}^*, y_i \rangle := \int_{\Omega_i} \psi_{i,q}'(g_i(\overline{z}_q)(x) - \varphi_i(x), y_i(x)) \mathrm{d}x$$

Property of the limit of the multipliers

Theorem (Gfrerer (4.2): Property of limit point)

Assume:

- *(P1) – (P4), (Q1) – (Q3)*
- $\hat{Z}$ *to be a Banach space continuously embedded in* Z
- $\hat{Y}$ *to be a Banach space continuously embedded in* $Y := \prod_{j=1}^{m} L^{r_j}(\Omega_j)$ *s.t.* $G(\hat{Z}) \subset \hat{Y}$.
- $(q^n) \subset Q \backslash \{\overline{q}\}$ *with* $q^n \to \overline{q}$
- $\langle y^*_{q^n}, y \rangle := \sum_{j=0}^{m} \langle l^*_{j,q^n}, y_j \rangle$
- $y^*_{q^n} \to_{w^*} y^* \in \hat{Y}^*$ *(convergence)*

Then:

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Assume:

- ...

Then:

- *y^* belongs to the normal cone of C at $G(\bar{z})$*
  (i.e., $\langle y^*, c - G(\bar{z})\rangle \leq 0 \; \forall c \in C$)
  where $C := \{c = (c_1, \dots, c_n) \in \hat{Y} \; : \; c_j \leq \varphi_j \text{ a.e.i. } \Omega_j \; \forall j\}$
  $G(z) := (g_1(z), \dots, g_n(z))$
- *$\bar{z}$ is the unique solution of*

$$\min_{z \in \hat{Z}_E} J(z) + \langle y^*, G(z)\rangle$$

That means: *y^* is a multiplier.*

Convergence of the multipliers

Theorem (Gfrerer (4.3): Convergence)

Assume:

- *(P1) – (P4), (Q1) – (Q3)*

(Q4) $0_Y \in \text{int}(G(\hat{Z}_E) - C)$

Then:

- $\limsup_{q \to \overline{q}} ||y_q^*||_{\hat{Y}^*} < \infty$

This implies the existence of a weak--convergent subsequence.*

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Convergence of the multipliers (cont.)

Theorem (Gfrerer (4.4): Convergence)

Assume:

- *the constraints can be partitioned* $C = C_1 \times C_2 \subset \hat{Y}_1 \times \hat{Y}_2 = \hat{Y}$ $G = (G_1, G_2)$ *where* $G_i : \hat{Z} \to \hat{Y}_i$ $(i = 1, 2)$ *such that*
  - $c_1 \in \operatorname{int} G_1(\hat{Z}_E)$ *for some* $c_1 \in C_1$ *(or more generally,* $0 \in \operatorname{int}_{\hat{Y}_1}(G_1(\hat{Z}_E) - C_1)$*) and*
  - $G_1(\tilde{z}) \in C_1,\ G_2(\tilde{z}) \in \operatorname{int}_{\hat{Y}_2} C_2$ *for some* $\tilde{z} \in \hat{Z}_E$

Then: *condition (Q4) is fulfilled.*

Application to model problem

- $\Omega \subset \mathbb{R}^d$ with $C^{1,1}$ boundary
- Assume $\overline{u} \in L^r(\Omega)$ with $r > \max\{d, 2\}$
  (can be ensured using box constraints)
- Choose $\hat{Z} := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \times L^r(\Omega)$
- Choose $\hat{Y} := L^r(\Omega) \times C(\text{cl}\Omega) \times C(\text{cl}\Omega)$
  (since y and ∇y are continuous on $\text{cl}\Omega$ for $y \in W^{2,r}(\Omega)$)
- $G(y, u) := (u, y, |\nabla y|_2)$
- $C := C_u \times C_y \times C_g = \{u \in L^r(\Omega) \ : \ u \le \varphi_u\} \times \{y \in C(\text{cl}\Omega) \ : \ y \le \varphi_y\} \times \{g \in C(\text{cl}\Omega) \ : \ g \le \varphi_g\}$
- (Q4) is fulfilled if there exists some feasible $(y, u) \in \hat{Z}$ with $y \in \text{int}\, C_y$, $|\nabla y|_2 \in \text{int}\, C_g$
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Thanks for your attention!

Literature

- H. Gfrerer: Generalized Penalty Methods for a Class of Convex Optimization Problems with Pointwise Inequality Constraints