# Seminar Numerical Analysis
# Infinite Dimensional Optimization

Institute of Computational Mathematics
Johannes Kepler University Linz, Austria

# Outline

1. First-order optimality conditions
   - Equality constraints
   - Inequality constraints
2. Solution methods
   - Semi-smooth Newton methods
   - Penalty methods

# Outline

# The finite dimensional case

$$(P_{eq}) \qquad\qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $J : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable
- Lagrangian: $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, \lambda) := J(z) + \lambda^T G(z)$$

### Theorem

*Assume*

- $\bar{z}$ *is a local minimizer for the problem* $(P_{eq})$
- $G'(\bar{z})$ *(the Jacobian of G at* $\bar{z}$*) has full rank m*

*Then there is a (unique) multiplier* $\bar{\lambda} \in \mathbb{R}^m$ *such that*

$$\nabla_z \mathcal{L}(\bar{z}, \bar{\lambda}) = 0.$$

# The finite dimensional case

$$(P_{eq}) \qquad\qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $J : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable
- Lagrangian: $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, \lambda) := J(z) + \lambda^T G(z)$$

**Theorem**

*Assume*

- *$\bar{z}$ is a local minimizer for the problem $(P_{eq})$*
- *$G'(\bar{z})$ (the Jacobian of G at $\bar{z}$) has full rank m*

*Then there is a (unique) multiplier $\bar{\lambda} \in \mathbb{R}^m$ such that*

$$\nabla_z \mathcal{L}(\bar{z}, \bar{\lambda}) = 0.$$

# The finite dimensional case

$$(P_{eq}) \qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $J : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable
- Lagrangian: $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, \lambda) := J(z) + \lambda^T G(z)$$

**Theorem**

*Assume*

- *$\bar{z}$ is a local minimizer for the problem $(P_{eq})$*
- *$G'(\bar{z})$ (the Jacobian of $G$ at $\bar{z}$) has full rank $m$*

*Then there is a (unique) multiplier $\bar{\lambda} \in \mathbb{R}^m$ such that*

$$\nabla_z \mathcal{L}(\bar{z}, \bar{\lambda}) = 0.$$

# The finite dimensional case

$$(P_{eq}) \qquad\qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $J : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable
- Lagrangian: $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, \lambda) := J(z) + \lambda^T G(z)$$

## Theorem

*Assume*

- *$\bar{z}$ is a local minimizer for the problem $(P_{eq})$*
- *$G'(\bar{z})$ (the Jacobian of $G$ at $\bar{z}$) has full rank $m$*

*Then there is a (unique) multiplier $\bar{\lambda} \in \mathbb{R}^m$ such that*

$$\nabla_z \mathcal{L}(\bar{z}, \bar{\lambda}) = 0.$$

# Solution method

- First-order necessary conditions are the base of a solution method.
- Define $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$,

$$(z, \lambda) \to F(z, \lambda) = \begin{pmatrix} \nabla_z \mathcal{L}(z, \lambda) \\ G(z) \end{pmatrix}$$

- A local minimizer $\bar{z}$ together with a multiplier $\bar{\lambda}$ (under full rank assumption on $G'(\bar{z})$) is solution of the nonlinear equation

$$F(z, \lambda) = 0$$

- If $F$ is continuously differentiable (i.e., $J$ and $G$ are twice continuously differentiable), then nonlinear equation $F(z, \lambda) = 0$ can be solved by (damped) *Newton method*
- $F'(\bar{z}, \bar{\lambda})$ is regular if some *second-order sufficient condition* is fulfilled.

# Solution method

- First-order necessary conditions are the base of a solution method.
- Define $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$,

$$(z, \lambda) \to F(z, \lambda) = \begin{pmatrix} \nabla_z \mathcal{L}(z, \lambda) \\ G(z) \end{pmatrix}$$

- A local minimizer $\bar{z}$ together with a multiplier $\bar{\lambda}$ (under full rank assumption on $G'(\bar{z})$) is solution of the nonlinear equation

$$F(z, \lambda) = 0$$

- If $F$ is continuously differentiable (i.e., $J$ and $G$ are twice continuously differentiable), then nonlinear equation $F(z, \lambda) = 0$ can be solved by (damped) *Newton method*
- $F'(\bar{z}, \bar{\lambda})$ is regular if some *second-order sufficient condition* is fulfilled.

# Solution method

- First-order necessary conditions are the base of a solution method.
- Define $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$,

$$(z, \lambda) \to F(z, \lambda) = \begin{pmatrix} \nabla_z \mathcal{L}(z, \lambda) \\ G(z) \end{pmatrix}$$

- A local minimizer $\bar{z}$ together with a multiplier $\bar{\lambda}$ (under full rank assumption on $G'(\bar{z})$) is solution of the nonlinear equation

$$F(z, \lambda) = 0$$

- If $F$ is continuously differentiable (i.e., $J$ and $G$ are twice continuously differentiable), then nonlinear equation $F(z, \lambda) = 0$ can be solved by (damped) *Newton method*
- $F'(\bar{z}, \bar{\lambda})$ is regular if some *second-order sufficient condition* is fulfilled.

# Solution method

- First-order necessary conditions are the base of a solution method.
- Define $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$,

$$(z, \lambda) \to F(z, \lambda) = \begin{pmatrix} \nabla_z \mathcal{L}(z, \lambda) \\ G(z) \end{pmatrix}$$

- A local minimizer $\bar{z}$ together with a multiplier $\bar{\lambda}$ (under full rank assumption on $G'(\bar{z})$) is solution of the nonlinear equation

$$F(z, \lambda) = 0$$

- If $F$ is continuously differentiable (i.e., $J$ and $G$ are twice continuously differentiable), then nonlinear equation $F(z, \lambda) = 0$ can be solved by (damped) *Newton method*
- $F'(\bar{z}, \bar{\lambda})$ is regular if some *second-order sufficient condition* is fulfilled.

# Solution method

- First-order necessary conditions are the base of a solution method.
- Define $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$,

$$(z, \lambda) \to F(z, \lambda) = \begin{pmatrix} \nabla_z \mathcal{L}(z, \lambda) \\ G(z) \end{pmatrix}$$

- A local minimizer $\bar{z}$ together with a multiplier $\bar{\lambda}$ (under full rank assumption on $G'(\bar{z})$) is solution of the nonlinear equation

$$F(z, \lambda) = 0$$

- If $F$ is continuously differentiable (i.e., $J$ and $G$ are twice continuously differentiable), then nonlinear equation $F(z, \lambda) = 0$ can be solved by (damped) *Newton method*
- $F'(\bar{z}, \bar{\lambda})$ is regular if some *second-order sufficient condition* is fulfilled.

# The infinite dimensional case

$$(P_{eq}) \qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $Z$, $V$ Banach spaces
- $J : Z \to \mathbb{R}$, $G : Z \to V$ continuously differentiable
- Lagrangian: $\mathcal{L} : Z \times V^* \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, v^*) := J(z) + \langle v^*, G(z) \rangle$$

**Theorem**

*Assume*

- *$\bar{z}$ is a local minimizer for the problem $(P_{eq})$*
- *The Fréchet derivative $DG(\bar{z}) \in L(Z, V)$ is surjective*

*Then there is a (unique) multiplier $\bar{v}^* \in V^*$ such that*

$$D_z \mathcal{L}(\bar{z}, \bar{v}^*) = 0.$$

# The infinite dimensional case

$$(P_{eq}) \qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $Z$, $V$ Banach spaces
- $J : Z \to \mathbb{R}$, $G : Z \to V$ continuously differentiable
- Lagrangian: $\mathcal{L} : Z \times V^* \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, v^*) := J(z) + \langle v^*, G(z) \rangle$$

**Theorem**

*Assume*

- *$\bar{z}$ is a local minimizer for the problem $(P_{eq})$*
- *The Fréchet derivative $DG(\bar{z}) \in L(Z, V)$ is surjective*

*Then there is a (unique) multiplier $\bar{v}^* \in V^*$ such that*

$$D_z \mathcal{L}(\bar{z}, \bar{v}^*) = 0.$$

# The infinite dimensional case

$$(P_{eq}) \qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $Z$, $V$ Banach spaces
- $J : Z \to \mathbb{R}$, $G : Z \to V$ continuously differentiable
- Lagrangian: $\mathcal{L} : Z \times V^* \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, v^*) := J(z) + \langle v^*, G(z) \rangle$$

**Theorem**

*Assume*

- *$\bar{z}$ is a local minimizer for the problem $(P_{eq})$*
- *The Fréchet derivative $DG(\bar{z}) \in L(Z, V)$ is surjective*

*Then there is a (unique) multiplier $\bar{v}^* \in V^*$ such that*

$$D_z\mathcal{L}(\bar{z}, \bar{v}^*) = 0.$$

# The infinite dimensional case

$$(P_{eq}) \qquad \min J(z) \quad \text{s.t.} \quad G(z) = 0$$

- $Z$, $V$ Banach spaces
- $J : Z \to \mathbb{R}$, $G : Z \to V$ continuously differentiable
- Lagrangian: $\mathcal{L} : Z \times V^* \to \mathbb{R}$,

$$(z, \lambda) \to \mathcal{L}(z, v^*) := J(z) + \langle v^*, G(z) \rangle$$

## Theorem

*Assume*

- *$\bar{z}$ is a local minimizer for the problem $(P_{eq})$*
- *The Fréchet derivative $DG(\bar{z}) \in L(Z, V)$ is surjective*

*Then there is a (unique) multiplier $\bar{v}^* \in V^*$ such that*

$$D_z \mathcal{L}(\bar{z}, \bar{v}^*) = 0.$$

# Lyusternik's Theorem

The first-order necessary conditions above are based on the following theorem

**Theorem (Lyusternik)**

*Assume*

- $G(\bar{z}) = 0$,
- $G'(\bar{z})$ *is surjective.*

*Then for every* $h \in Z$ *with* $G'(\bar{z})h = 0$ *there is some* $z(h) \in Z$ *with* $G(z(h)) = 0$ *and*

$$\|z(h) - (\bar{z} + h)\| = o(\|h\|)$$

In fact, with some small modifications of the proof of Lyusternik's Theorem one can derive the following regularity result ($\operatorname{dist}(z, A) := \inf\{\|z - a\| \ : \ a \in A\}$, $\operatorname{dist}(z, \emptyset) := \infty$):

## Theorem

*Under the assumptions of Lyusternik's theorem there is some positive real $\kappa$ and some neighborhood $N \subset Z \times V$ of $(\bar{z}, 0)$ such that*

$$\operatorname{dist}(z, G^{-1}(v)) \leq \kappa \|G(z) - v\|, \ \forall (z, v) \in N$$

i.e.

- For every $v \in V$ sufficiently close to 0 the equation $G(z) = v$ has at least one solution near $\bar{z}$.
- For every $v \in V$ sufficiently close to 0 and for every $z'$ sufficiently close to $\bar{z}$ the distance between $z'$ and the solution set of $G(z) = v$ is proportionally bounded by the norm of the residual

In fact, with some small modifications of the proof of Lyusternik's Theorem one can derive the following regularity result ($\operatorname{dist}(z, A) := \inf\{\|z - a\| \ : \ a \in A\}$, $\operatorname{dist}(z, \emptyset) := \infty$):

**Theorem**

*Under the assumptions of Lyusternik's theorem there is some positive real $\kappa$ and some neighborhood $N \subset Z \times V$ of $(\bar{z}, 0)$ such that*

$$\operatorname{dist}(z, G^{-1}(v)) \leq \kappa \|G(z) - v\|, \ \forall (z, v) \in N$$

i.e.

- For every $v \in V$ sufficiently close to 0 the equation $G(z) = v$ has at least one solution near $\bar{z}$.
- For every $v \in V$ sufficiently close to 0 and for every $z'$ sufficiently close to $\bar{z}$ the distance between $z'$ and the solution set of $G(z) = v$ is proportionally bounded by the norm of the residual

In fact, with some small modifications of the proof of Lyusternik's Theorem one can derive the following regularity result ($\operatorname{dist}(z, A) := \inf\{\|z - a\| \ : \ a \in A\}$, $\operatorname{dist}(z, \emptyset) := \infty$):

**Theorem**

*Under the assumptions of Lyusternik's theorem there is some positive real $\kappa$ and some neighborhood $N \subset Z \times V$ of $(\bar{z}, 0)$ such that*

$$\operatorname{dist}(z, G^{-1}(v)) \leq \kappa \|G(z) - v\|, \ \forall (z, v) \in N$$

i.e.

- For every $v \in V$ sufficiently close to 0 the equation $G(z) = v$ has at least one solution near $\bar{z}$.
- For every $v \in V$ sufficiently close to 0 and for every $z'$ sufficiently close to $\bar{z}$ the distance between $z'$ and the solution set of $G(z) = v$ is proportionally bounded by the norm of the residual

# Outline

1. First-order optimality conditions
   - Equality constraints
   - Inequality constraints

2. Solution methods
   - Semi-smooth Newton methods
   - Penalty methods

# Problems with abstract inequality constraints

$$(P_{ineq}) \qquad \min J(z) \quad s.t. \quad 0 \in G(z)$$

- $Z$, $V$ Banach spaces, $J : Z \to \mathbb{R}$
- $G : Z \rightrightarrows V$ is a *multifunctuion* (*set-valued function, multi-valued function*), i.e. $G(z)$ is a subset of $V$
- Important special case:

$$G(z) = g(z) - C,$$

where $g : Z \to V$ is a mapping, $C \subset V$ is a (closed convex) set.
- Structure of optimality conditions:

$$\exists \bar{v}^* \in N_C(g(\bar{z})) : DJ(\bar{z}) + Dg(\bar{z})^* v^* = 0,$$

where $N_C(g(\bar{z})) := \{v^* \in V^* \ : \ \langle v^*, c - g(\bar{z})\rangle \leq 0, \forall c \in C\}$

# Problems with abstract inequality constraints

$$(P_{ineq}) \qquad \min J(z) \quad s.t. \quad 0 \in G(z)$$

- $Z$, $V$ Banach spaces, $J : Z \to \mathbb{R}$
- $G : Z \rightrightarrows V$ is a *multifunctuion* (*set-valued function, multi-valued function*), i.e. $G(z)$ is a subset of $V$
- Important special case:

$$G(z) = g(z) - C,$$

  where $g : Z \to V$ is a mapping, $C \subset V$ is a (closed convex) set.
- Structure of optimality conditions:

$$\exists \bar{v}^* \in N_C(g(\bar{z})) : DJ(\bar{z}) + Dg(\bar{z})^* v^* = 0,$$

  where $N_C(g(\bar{z})) := \{v^* \in V^* \ : \ \langle v^*, c - g(\bar{z})\rangle \leq 0, \forall c \in C\}$

# Problems with abstract inequality constraints

$$(P_{ineq}) \qquad \min J(z) \quad s.t. \quad 0 \in G(z)$$

- $Z$, $V$ Banach spaces, $J : Z \to \mathbb{R}$
- $G : Z \rightrightarrows V$ is a *multifunctuion* (*set-valued function, multi-valued function*), i.e. $G(z)$ is a subset of $V$
- Important special case:

$$G(z) = g(z) - C,$$

  where $g : Z \to V$ is a mapping, $C \subset V$ is a (closed convex) set.
- Structure of optimality conditions:

$$\exists \bar{v}^* \in N_C(g(\bar{z})) : DJ(\bar{z}) + Dg(\bar{z})^* v^* = 0,$$

  where $N_C(g(\bar{z})) := \{v^* \in V^* \ : \ \langle v^*, c - g(\bar{z})\rangle \leq 0, \forall c \in C\}$

# Problems with abstract inequality constraints

$$(P_{ineq}) \qquad \min J(z) \quad s.t. \quad 0 \in G(z)$$

- $Z$, $V$ Banach spaces, $J : Z \to \mathbb{R}$
- $G : Z \rightrightarrows V$ is a *multifunctuion* (*set-valued function, multi-valued function*), i.e. $G(z)$ is a subset of $V$
- Important special case:

$$G(z) = g(z) - C,$$

  where $g : Z \to V$ is a mapping, $C \subset V$ is a (closed convex) set.
- Structure of optimality conditions:

$$\exists \bar{v}^* \in N_C(g(\bar{z})) : DJ(\bar{z}) + Dg(\bar{z})^* v^* = 0,$$

  where $N_C(g(\bar{z})) := \{v^* \in V^* \, : \, \langle v^*, c - g(\bar{z})\rangle \leq 0, \forall c \in C\}$

# Problems with abstract inequality constraints

$$(P_{ineq}) \qquad \min J(z) \quad s.t. \quad 0 \in G(z)$$

- $Z$, $V$ Banach spaces, $J : Z \to \mathbb{R}$
- $G : Z \rightrightarrows V$ is a *multifunctuion* (*set-valued function, multi-valued function*), i.e. $G(z)$ is a subset of $V$
- Important special case:

$$G(z) = g(z) - C,$$

  where $g : Z \to V$ is a mapping, $C \subset V$ is a (closed convex) set.
- Structure of optimality conditions:

$$\exists \bar{v}^* \in N_C(g(\bar{z})) : DJ(\bar{z}) + Dg(\bar{z})^* v^* = 0,$$

  where $N_C(g(\bar{z})) := \{v^* \in V^* \ : \ \langle v^*, c - g(\bar{z})\rangle \leq 0, \forall c \in C\}$

# Example: Obstacle problem

## Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

## Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega f y \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega f y \, d\omega$

# Example: Obstacle problem

Problem formulation

- An elastic membrane is attached to a flat wireframe.
- Forces are acting in vertical direction.
- A rigid body (*obstacle*) is placed under the membrane.
- We look for the deflexion of the membrane

Mathematical model

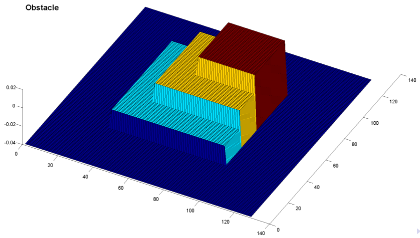
- $\Omega \subset \mathbb{R}^2$: domain enclosed by the wireframe
- $y(x)$, $x \in \Omega$: deflexion of the deformed membrane
- Membrane is fixed at the wireframe $\Rightarrow y(x) = 0$, $x \in \partial\Omega$
- Obstacle is given by $\varphi \in L^2(\Omega)$
- Forces are given by $f \in L^2(\Omega)$
- Total energy of deformed membrane: $J(y) = P(y) - E(y)$
- Potential energy: $P(y) \approx \frac{1}{2} \int_\Omega |\nabla y|^2 \, d\omega$
- Energy due to external forces: $E(y) = \int_\Omega fy \, d\omega$

# Model of the obstacle problem

The equilibrium is given by minimizing the energy functional $J(y)$, when the deflection of the membrane is restricted from below by the obstacle:

$$\begin{aligned} \min\ J(y) &= \frac{1}{2}\int_\Omega |\nabla y|^2\, d\omega - \int_\Omega fy\, d\omega \\ \text{s.t.} \quad y &= 0 \text{ on } \partial\Omega \\ y &\geq \varphi \text{ a.e. in } \Omega \end{aligned}$$

- e.g. $\Omega = (0,1)^2$
- $f = 3$

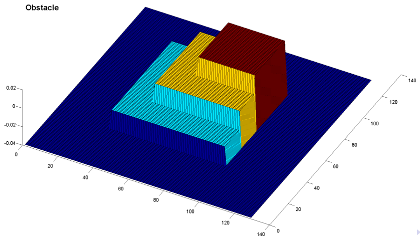


# Model of the obstacle problem

The equilibrium is given by minimizing the energy functional $J(y)$, when the deflection of the membrane is restricted from below by the obstacle:

$$\begin{aligned} \min\ J(y) &= \frac{1}{2}\int_\Omega |\nabla y|^2\, d\omega - \int_\Omega f y\, d\omega \\ \text{s.t.} \quad y &= 0 \text{ on } \partial\Omega \\ y &\geq \varphi \text{ a.e. in } \Omega \end{aligned}$$

- e.g. $\Omega = (0,1)^2$
- $f \equiv 3$

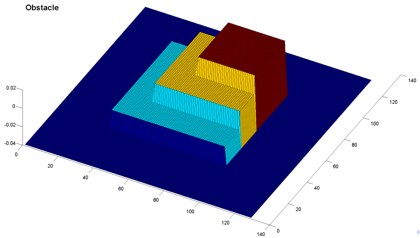


# Model of the obstacle problem

The equilibrium is given by minimizing the energy functional $J(y)$, when the deflection of the membrane is restricted from below by the obstacle:

$$\begin{aligned} \min \; J(y) &= \frac{1}{2}\int_\Omega |\nabla y|^2 \, d\omega - \int_\Omega fy \, d\omega \\ \text{s.t.} \quad y &= 0 \text{ on } \partial\Omega \\ y &\geq \varphi \text{ a.e. in } \Omega \end{aligned}$$

- e.g. $\Omega = (0,1)^2$
- $f \equiv 3$

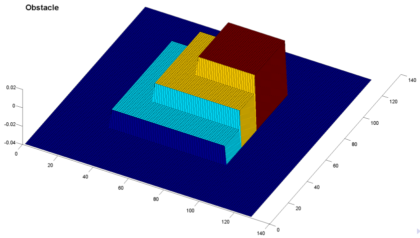


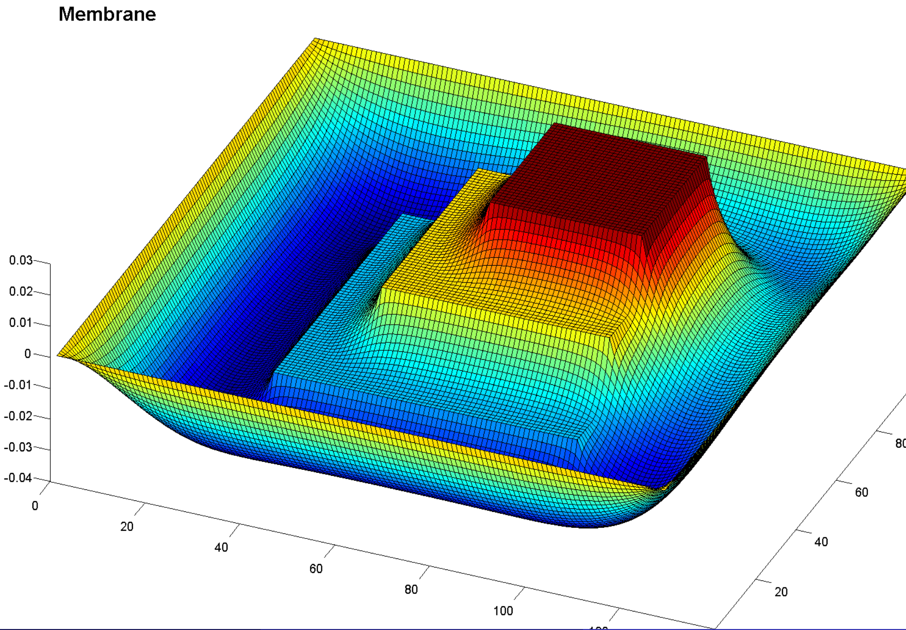
# Model of the obstacle problem

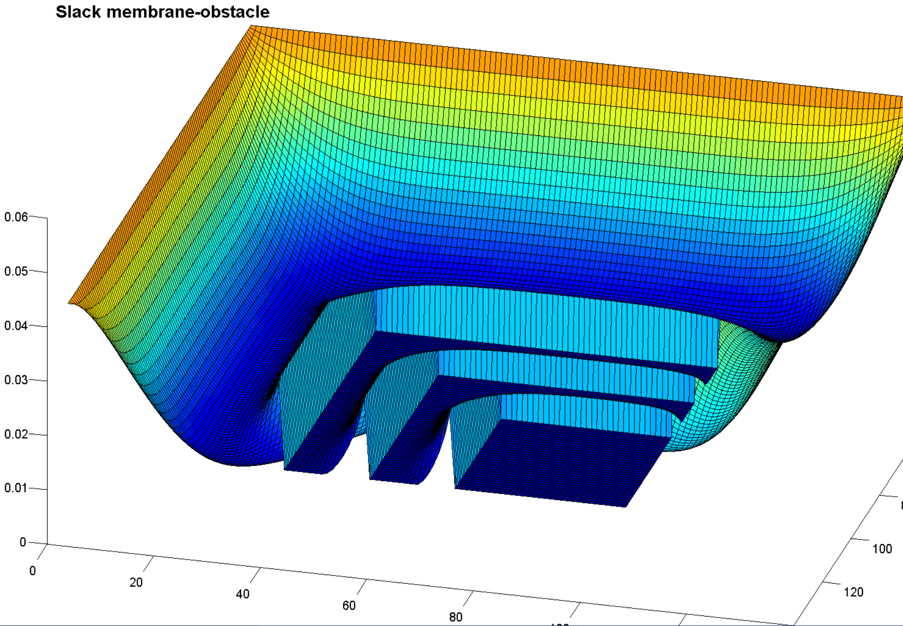
The equilibrium is given by minimizing the energy functional $J(y)$, when the deflection of the membrane is restricted from below by the obstacle:

$$\begin{aligned} \min \ J(y) &= \frac{1}{2}\int_\Omega |\nabla y|^2 \, d\omega - \int_\Omega fy \, d\omega \\ \text{s.t.} \qquad y &= 0 \text{ on } \partial\Omega \\ y &\geq \varphi \text{ a.e. in } \Omega \end{aligned}$$

- e.g. $\Omega = (0,1)^2$
- $f \equiv 3$




Membrane
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0
20
40
60
80
100
20
40
60
80


Slack membrane-obstacle
0.06
0.05
0.04
0.03
0.02
0.01
0
0
20
40
60
80
100
120

# Formulation as ($P_{ineq}$)

- $Z = H_0^1(\Omega)$, $J(z) = \frac{1}{2}\int_\Omega |\nabla z|^2 \, d\omega - \int_\Omega fz \, d\omega$
- Let $i : H_0^1(\Omega) \to L^2(\Omega)$ denote the canonical injection, $K = \{u \in L^2(\Omega) \, : \, u \geq \varphi \text{ a.e.}\}$
- 2 equivalent possibilities for $G$, $V$
  1. $V = L^2(\Omega)$, $g = i$, $C = K$, i.e. $G(z) = i(z) - K$.
  2. $V = H_0^1(\Omega)$, $g = id_{H_0^1(\Omega)}$, $C = i^{-1}(K)$, i.e. $G(z) = z - i^{-1}(K)$.
- First-order necessary conditions are only possible in the second case.
- In order that first-order necessary conditions are fulfilled at a solution $\bar{z}$ of $P_{ineq}$ some *constraint qualification* condition (CQ) has to be fulfilled .

# Formulation as ($P_{ineq}$)

- $Z = H_0^1(\Omega)$, $J(z) = \frac{1}{2}\int_\Omega |\nabla z|^2 \, d\omega - \int_\Omega fz \, d\omega$
- Let $i : H_0^1(\Omega) \to L^2(\Omega)$ denote the canonical injection, $K = \{u \in L^2(\Omega) \, : \, u \geq \varphi \text{ a.e.}\}$
- 2 equivalent possibilities for $G$, $V$
  1. $V = L^2(\Omega)$, $g = i$, $C = K$, i.e. $G(z) = i(z) - K$.
  2. $V = H_0^1(\Omega)$, $g = id_{H_0^1(\Omega)}$, $C = i^{-1}(K)$, i.e. $G(z) = z - i^{-1}(K)$.
- First-order necessary conditions are only possible in the second case.
- In order that first-order necessary conditions are fulfilled at a solution $\bar{z}$ of $P_{ineq}$ some *constraint qualification* condition (CQ) has to be fulfilled .

## Formulation as ($P_{ineq}$)

- $Z = H_0^1(\Omega)$, $J(z) = \frac{1}{2}\int_\Omega |\nabla z|^2 \, d\omega - \int_\Omega fz \, d\omega$
- Let $i : H_0^1(\Omega) \to L^2(\Omega)$ denote the canonical injection, $K = \{u \in L^2(\Omega) \, : \, u \geq \varphi \text{ a.e.}\}$
- 2 equivalent possibilities for $G$, $V$
  1. $V = L^2(\Omega)$, $g = i$, $C = K$, i.e. $G(z) = i(z) - K$.
  2. $V = H_0^1(\Omega)$, $g = id_{H_0^1(\Omega)}$, $C = i^{-1}(K)$, i.e. $G(z) = z - i^{-1}(K)$.
- First-order necessary conditions are only possible in the second case.
- In order that first-order necessary conditions are fulfilled at a solution $\bar{z}$ of $P_{ineq}$ some *constraint qualification* condition (CQ) has to be fulfilled .

# Formulation as ($P_{ineq}$)

- $Z = H_0^1(\Omega)$, $J(z) = \frac{1}{2}\int_\Omega |\nabla z|^2 \, d\omega - \int_\Omega fz \, d\omega$
- Let $i : H_0^1(\Omega) \to L^2(\Omega)$ denote the canonical injection, $K = \{u \in L^2(\Omega) \, : \, u \geq \varphi \text{ a.e.}\}$
- 2 equivalent possibilities for $G$, $V$
  1. $V = L^2(\Omega)$, $g = i$, $C = K$, i.e. $G(z) = i(z) - K$.
  2. $V = H_0^1(\Omega)$, $g = id_{H_0^1(\Omega)}$, $C = i^{-1}(K)$, i.e. $G(z) = z - i^{-1}(K)$.
- First-order necessary conditions are only possible in the second case.
- In order that first-order necessary conditions are fulfilled at a solution $\bar{z}$ of $P_{ineq}$ some *constraint qualification* condition (CQ) has to be fulfilled .

# Formulation as ($P_{ineq}$)

- $Z = H_0^1(\Omega)$, $J(z) = \frac{1}{2}\int_\Omega |\nabla z|^2 \, d\omega - \int_\Omega fz \, d\omega$
- Let $i : H_0^1(\Omega) \to L^2(\Omega)$ denote the canonical injection, $K = \{u \in L^2(\Omega) \, : \, u \geq \varphi \text{ a.e.}\}$
- 2 equivalent possibilities for $G$, $V$
  1. $V = L^2(\Omega)$, $g = i$, $C = K$, i.e. $G(z) = i(z) - K$.
  2. $V = H_0^1(\Omega)$, $g = id_{H_0^1(\Omega)}$, $C = i^{-1}(K)$, i.e. $G(z) = z - i^{-1}(K)$.
- First-order necessary conditions are only possible in the second case.
- In order that first-order necessary conditions are fulfilled at a solution $\bar{z}$ of $P_{ineq}$ some *constraint qualification* condition (CQ) has to be fulfilled .

# Metric regularity

## Definition

Let $G : Z \rightrightarrows V$ be a multifunction, $0 \in G(\bar{z})$. $G$ is called *metrically regular near* $(\bar{z}, 0)$ iff there are some positive real $\kappa$ and some neighborhood $N \subset Z \times V$ of $(\bar{z}, 0)$ such that

$$\operatorname{dist}(z, G^{-1}(v)) \leq \kappa \operatorname{dist}(v, G(z)), \ \forall (z, v) \in N$$

- Metric regularity provides a CQ.
- Besides its importance for optimality conditions it is a fundamental stability concept.
- Metric regularity guarantees the solvability of the constraints under small perturbations and the continuity of solutions in dependence of the perturbations

# Metric regularity

**Definition**

Let $G : Z \rightrightarrows V$ be a multifunction, $0 \in G(\bar{z})$. $G$ is called *metrically regular near* $(\bar{z}, 0)$ iff there are some positive real $\kappa$ and some neighborhood $N \subset Z \times V$ of $(\bar{z}, 0)$ such that

$$\operatorname{dist}(z, G^{-1}(v)) \leq \kappa \operatorname{dist}(v, G(z)), \ \forall (z, v) \in N$$

- Metric regularity provides a CQ.
- Besides its importance for optimality conditions it is a fundamental stability concept.
- Metric regularity guarantees the solvability of the constraints under small perturbations and the continuity of solutions in dependence of the perturbations

# Metric regularity

**Definition**

Let $G : Z \rightrightarrows V$ be a multifunction, $0 \in G(\bar{z})$. $G$ is called *metrically regular near* $(\bar{z}, 0)$ iff there are some positive real $\kappa$ and some neighborhood $N \subset Z \times V$ of $(\bar{z}, 0)$ such that

$$\operatorname{dist}(z, G^{-1}(v)) \leq \kappa \operatorname{dist}(v, G(z)), \ \forall (z, v) \in N$$

- Metric regularity provides a CQ.
- Besides its importance for optimality conditions it is a fundamental stability concept.
- Metric regularity guarantees the solvability of the constraints under small perturbations and the continuity of solutions in dependence of the perturbations

# Metric regularity

## Definition

Let $G : Z \rightrightarrows V$ be a multifunction, $0 \in G(\bar{z})$. $G$ is called *metrically regular near* $(\bar{z}, 0)$ iff there are some positive real $\kappa$ and some neighborhood $N \subset Z \times V$ of $(\bar{z}, 0)$ such that

$$\operatorname{dist}(z, G^{-1}(v)) \leq \kappa \operatorname{dist}(v, G(z)), \ \forall (z, v) \in N$$

- Metric regularity provides a CQ.
- Besides its importance for optimality conditions it is a fundamental stability concept.
- Metric regularity guarantees the solvability of the constraints under small perturbations and the continuity of solutions in dependence of the perturbations

# Outline

1. First-order optimality conditions
   - Equality constraints
   - Inequality constraints

2. Solution methods
   - Semi-smooth Newton methods
   - Penalty methods

# Semi-Smooth Newton method

- Method for solving first-order necessary conditions.
- The normal cone of $C$ is given by a variational inequality. Sometimes this variational inequality can be reformulated as a non-smooth equation.
- Semi-smooth Newton methods are intended for solving special non-smooth equations.

# Semi-Smooth Newton method

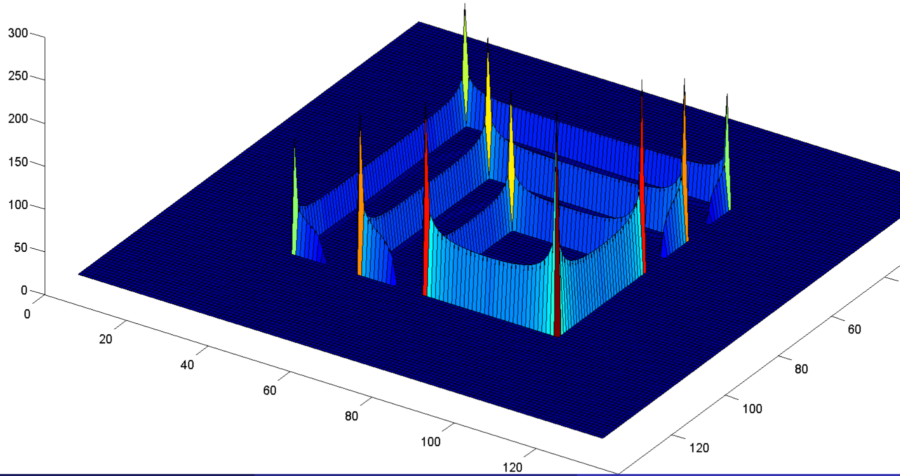
- Method for solving first-order necessary conditions.
- The normal cone of $C$ is given by a variational inequality. Sometimes this variational inequality can be reformulated as a non-smooth equation.
- Semi-smooth Newton methods are intended for solving special non-smooth equations.

# Semi-Smooth Newton method

- Method for solving first-order necessary conditions.
- The normal cone of $C$ is given by a variational inequality. Sometimes this variational inequality can be reformulated as a non-smooth equation.
- Semi-smooth Newton methods are intended for solving special non-smooth equations.

# Outline

Multiplier



# Penalty methods

- First-order conditions for obstacle problem cannot be solved directly by semi-smooth Newton methods, since multiplier $\bar{v}^* \in H^{-1}(\Omega)$
- Penalty methods do not rely on first-order conditions
- Original problem is replaced by a sequence of unconstrained (or equality constrained) problems.

# Penalty methods

- First-order conditions for obstacle problem cannot be solved directly by semi-smooth Newton methods, since multiplier $\bar{v}^* \in H^{-1}(\Omega)$
- Penalty methods do not rely on first-order conditions
- Original problem is replaced by a sequence of unconstrained (or equality constrained) problems.

# Penalty methods

- First-order conditions for obstacle problem cannot be solved directly by semi-smooth Newton methods, since multiplier $\bar{v}^* \in H^{-1}(\Omega)$
- Penalty methods do not rely on first-order conditions
- Original problem is replaced by a sequence of unconstrained (or equality constrained) problems.