SupInf condition, 1st order optimality conditions

Arpan Ghosh

—

24-11-2009

Definition 1 : Let $\Psi : X \rightrightarrows Y$ be a set valued map, $\bar{y} \in \Psi(\bar{x})$. The multifunction Ψ is called metrically regular near $(\bar{x}, \bar{y})$ if there are neighborhoods $N_{\bar{x}}$, $N_{\bar{y}}$ of $\bar{x}, \bar{y}$ respectively, and some $k > 0$ such that

$$d(x, \Psi^{-1}(y)) \leq kd(y, \Psi(x)) \forall (x, y) \in N_{\bar{x}} \times N_{\bar{y}} \qquad (1)$$

The lower bound of such k will be called the constant of metric regularity of Ψ near $(\bar{x}, \bar{y})$ and denoted by $Reg\Psi(\bar{x}, \bar{y})$.

Definition 2. For a set $D \subset Z$, the support function $\sigma_D : Z^* \to \mathbb{R}$ is defined by

$$\sigma_D(z^*) := \sup_{z \in D} \langle z^*, z \rangle$$

Definition 2. For a set $D \subset Z$, the support function $\sigma_D : Z^* \to \mathbb{R}$ is defined by

$$\sigma_D(z^*) := \sup_{z \in D} \langle z^*, z \rangle$$

Lemma 1 : Let D be a closed convex subset of a normed space Z and let $\hat{d}_D : Z \to \mathbb{R} \cup \{\pm\infty\}$ be given by

$$\hat{d}_D(z) := \sup_{z^* \in S_{Z^*}} \{\langle z^*, z \rangle - \sigma_D(z^*)\}.$$

Then $\forall z \in Z$ we have

$$\hat{d}_D(z) = \begin{cases} d(z, D) & \textit{if} \quad z \notin D \\ -\max(\rho : z + \rho B_Z \subset D) & \textit{if} \quad z \in D \end{cases} \tag{2}$$

Assumptions

- $\Gamma : X \rightrightarrows Y$ of the form $\Gamma(x) = h(x) - C$ where X and Y are Banach spaces, $h : X \to Y$ is an arbitrary mapping and $C \subset Y$ is closed and convex.

Assumptions

- $\Gamma : X \rightrightarrows Y$ of the form $\Gamma(x) = h(x) - C$ where X and Y are Banach spaces, $h : X \to Y$ is an arbitrary mapping and $C \subset Y$ is closed and convex.
- Let $h(\bar{x}) \in C$. Let h is strictly differentiable at $\bar{x}$, i.e., h is Fréchet differentiable at $\bar{x}$ and the derivative $h'(\bar{x})$ satisfies

$$\lim_{x,x' \to \bar{x}} \frac{\|h(x) - h(x') - h'(\bar{x})(x - x')\|}{\|x - x'\|} = 0,$$

Theorem 1. The following are equivalently:

1. The multifunction Γ is metrically regular near $(\bar{x}, 0)$.
2. The linearization
$$\Psi(x) := h(\bar{x}) + h'(\bar{x})(x - x') - C$$
is metrically regular near $(\bar{x}, 0)$.
3. There holds $0 \in int(h(\bar{x}) + h'(\bar{x})X - C)$.

Theorem 1. The following are equivalently:

1. The multifunction Γ is metrically regular near $(\bar{x}, 0)$.

2. The linearization
$$\Psi(x) := h(\bar{x}) + h'(\bar{x})(x - x') - C$$
is metrically regular near $(\bar{x}, 0)$.

3. There holds $0 \in int(h(\bar{x}) + h'(\bar{x})X - C)$.

- There is some $\tilde{\kappa} \geq 0$ such that $0 \in int(h(\bar{x}) + \tilde{\kappa} h'(\bar{x}) B_X - C)$

Theorem 1. The following are equivalently:

1. The multifunction Γ is metrically regular near $(\bar{x}, 0)$.
2. The linearization
$$\Psi(x) := h(\bar{x}) + h'(\bar{x})(x - x') - C$$
is metrically regular near $(\bar{x}, 0)$.
3. There holds $0 \in int(h(\bar{x}) + h'(\bar{x})X - C)$.

- There is some $\tilde{\kappa} \geq 0$ such that $0 \in int(h(\bar{x}) + \tilde{\kappa} h'(\bar{x})B_X - C)$
- $\forall \kappa > 0$, $0 \in int(h(\bar{x}) + \kappa h'(\bar{x})B_X - C)$

Definition 3. The function $\hat{d}_C : Y \times L(X, Y) \times \mathbb{R} \to \mathbb{R}$ is given by

$$\hat{d}_C(y, A, \kappa) := \sup_{y^* \in S_{Y^*}} \{\langle y^*, y\rangle - \sigma_C(y^*) - \kappa \|A^* y^*\|\}$$

Definition 3. The function $\hat{d}_C : Y \times L(X,Y) \times \mathbb{R} \to \mathbb{R}$ is given by

$$\hat{d}_C(y, A, \kappa) := \sup_{y^* \in S_{Y^*}} \{\langle y^*, y\rangle - \sigma_C(y^*) - \kappa \|A^* y^*\|\}$$

Proposition 1. For each $A \in L(X,Y)$, $\forall y \in Y$ and $\forall \kappa \geq 0$ let $D_C(y, A, \kappa) := y + \kappa A B_X - C$. Then

$$\hat{d}_C(y, A, \kappa) = \begin{cases} d(0, D_C(y, A, \kappa)) & if \quad 0 \notin clD_C(y, A, \kappa) \\ 0 & if \quad 0 \in bdD_C(y, A, \kappa) \\ -sup(\rho : \rho B_Y \subset D_C(y, A, \kappa) & if \quad 0 \in intD_C(y, A, \kappa) \end{cases} \tag{3}$$

SupInf Condition

Theorem 2. Γ is metrically regular near $(\bar{x}, 0)$ iff

$$\hat{d}_C(h(\bar{x}), h'(\bar{x}), \tilde{\kappa}) < 0 \text{ for some } \tilde{\kappa} \geq 0.$$

Lemma 2. Let $A \in L(X, Y)$, $y \in Y$, and $\kappa \geq 0$ be such that $\hat{d}_C(y, A, \kappa) < 0$. Then

$$d(0, A^{-1}(C - y)) \leq \frac{\kappa}{d(y, C) - \hat{d}_C(y, A, \kappa)} d(y, C)$$

Lemma 3. For given $A \in L(X, Y)$, $\tilde{y} \in C$, $\tilde{x} \in X$ if the set-valued map $\Psi(x) := \tilde{y} + A(x - \tilde{x}) - C$, is metrically regular near $(\tilde{x}, 0)$, then

$$Reg\Psi(\tilde{x}, 0) = \lim_{\kappa \to 0_+} \frac{\kappa}{-\hat{d}_C(\tilde{y}, A, \kappa)} = \inf_{\kappa > 0} \frac{\kappa}{-\hat{d}_C(\tilde{y}, A, \kappa)}$$

We consider the following problem

$$\text{(P)} \qquad \text{L-minimize } f(x) \text{ subject to } g(x) \in K,$$

where $f : X \to U$ and $g : X \to V$ are mappings from the Banach space X into other Banach spaces U and V, $K \subset V$ is a closed convex set and $L \subset U$ is a closed convex cone with none empty interior i.e. $intL \neq \emptyset$

Definition 4. An element $\bar{x} \in X$ is called a local weak minimizer for (P) if $g(\bar{x}) \in K$ and if there exists a neighborhood N of $\bar{x}$ such that for each $x \in N$ with $g(x) \in K$, one has $f(x) - f(\bar{x}) \notin$ -int$L$.

Definition 4. An element $\bar{x} \in X$ is called a local weak minimizer for (P) if $g(\bar{x}) \in K$ and if there exists a neighborhood N of $\bar{x}$ such that for each $x \in N$ with $g(x) \in K$, one has $f(x) - f(\bar{x}) \notin$ -int$L$.

- $h : X \to U \times V$ defined by $h(x) := (f(x) - f(\bar{x}), g(x))$
  h is strictly differentiable.

Definition 4. An element $\bar{x} \in X$ is called a local weak minimizer for (P) if $g(\bar{x}) \in K$ and if there exists a neighborhood N of $\bar{x}$ such that for each $x \in N$ with $g(x) \in K$, one has $f(x) - f(\bar{x}) \notin$ -int$L$.

- $h : X \to U \times V$ defined by $h(x) := (f(x) - f(\bar{x}), g(x))$ h is strictly differentiable.
- $C := (-L) \times K, Y := U \times V$

Definition 4. An element $\bar{x} \in X$ is called a local weak minimizer for (P) if $g(\bar{x}) \in K$ and if there exists a neighborhood N of $\bar{x}$ such that for each $x \in N$ with $g(x) \in K$, one has $f(x) - f(\bar{x}) \notin$ -int$L$.

- $h : X \to U \times V$ defined by $h(x) := (f(x) - f(\bar{x}), g(x))$
  h is strictly differentiable.
- $C := (-L) \times K, Y := U \times V$
- The multifunction $\Gamma : X \rightrightarrows Y$, is given by $\Gamma(x) := h(x) - C$

Conditions

If $\bar{x}$ is a local weak minimizer of problem (P)

If $\bar{x}$ is a local weak minimizer of problem (P)

$\Rightarrow \Gamma$ is not metrically regular near $(\bar{x}, 0)$

Conditions

If $\bar{x}$ is a local weak minimizer of problem (P)

$\Rightarrow \Gamma$ is not metrically regular near $(\bar{x}, 0)$

$\Leftrightarrow \hat{d}_C(h(\bar{x}), h'(\bar{x}), 1) :=$
$\sup_{y^* \in S_{Y^*}} \{\langle y^*, h(\bar{x})\rangle - \sigma_C(y^*) - \|h'(\bar{x})^* y^*\|\} \geq 0$

If $\bar{x}$ is a local weak minimizer of problem (P)

$\Rightarrow \Gamma$ is not metrically regular near $(\bar{x}, 0)$

$\Leftrightarrow \hat{d}_C(h(\bar{x}), h'(\bar{x}), 1) :=$
$\sup_{y^* \in S_{Y^*}} \{\langle y^*, h(\bar{x})\rangle - \sigma_C(y^*) - \|h'(\bar{x})^* y^*\|\} \geq 0$

$\Leftrightarrow \exists$ a sequence $(y_n^*) \subset S_{Y^*}$ such that

$$\lim_{n\to\infty} \{\sigma_C(y_n^*) - \langle y_n^*, h(\bar{x})\rangle\} = 0, \lim_{n\to\infty} \|h'(\bar{x})^* y_n^*\| = 0 \tag{4}$$

Fritz-John type optimality conditions:

- $$f'(\bar{x})^* u^* + g'(\bar{x})^* v^* = 0, 0 \neq (u^*, v^*) \in L^* \times N_K(g(\bar{x})) \subset U^* \times V^* \tag{5}$$

  where $L^* := \{u^* \in U^* : \langle u^*, u \rangle \geq 0 \forall u \in L\}$ is the dual cone of the cone L and $N_K(z) := \{v^* \in V^* : \langle v^*, v - z \rangle \leq 0 \forall v \in K\}$ is the normal cone to K at z.

Fritz-John type optimality conditions:

-
$$f'(\bar{x})^* u^* + g'(\bar{x})^* v^* = 0, 0 \neq (u^*, v^*) \in L^* \times N_K(g(\bar{x})) \subset U^* \times V^* \tag{5}$$
where $L^* := \{u^* \in U^* : \langle u^*, u \rangle \geq 0 \forall u \in L\}$ is the dual cone of the cone L and $N_K(z) := \{v^* \in V^* : \langle v^*, v - z \rangle \leq 0 \forall v \in K\}$ is the normal cone to K at z.

- Or in shorter form with $y^* := (u^*, v^*)$:
$$h'(\bar{x})^* y_n^* = 0, y^* \in N_C(h(\bar{x})), y^* \neq 0 \tag{6}$$

Kuhn-Tucker conditions: Kuhn-Tucker conditions hold if the Fritz-John conditions (5) hold with $u^* \neq 0$.

Kuhn-Tucker conditions: Kuhn-Tucker conditions hold if the Fritz-John conditions (5) hold with $u^* \neq 0$.

Let Λ_{FJ} denote the set of multipliers y^* satisfying the Fritz-John conditions (6).

Let Λ_{KT} denote the set of multipliers y^* satisfying the Kuhn-Tucker conditions.

Proposition 2. Assume that $\bar{x}$ is nondegenerate and let $(y_n^*) \subset S_{Y^*}$ be a sequence satisfying (4). Then $\forall$ weak* accumulation point $\bar{y}^*$ of (y_n^*), $\bar{y}^* \in \Lambda_{FJ} \cap B_{Y^*}$.

- In the case of scalar minimization the commonly used set of Lagrange multipliers is given by $\{v^* \in V^* : (1, v^*) \in \Lambda_{FJ}\}$
- Robinson's constraint qualification holds if and only if $(0, 1) \in int(h(\bar{x}) + h'(\bar{x})X - C)$
- **Theorem 3.** Let $\Lambda_{FJ} \neq \emptyset$ and assume that Robinson's constrain qualification $0 \in int(g(\bar{x}) + g'(\bar{x})X - K)$ is fulfilled. Then one has $\Lambda_{FJ} = \Lambda_{KT}$.

Thank You!!