Metric Regularity - Examples and Applications

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Outline

The problem

$$(P) \qquad \min_{z \in Z} J(z)$$
$$\text{s.t.} \quad 0 \in G(z)$$

- Z, V Banach spaces;
- $J : Z \to \mathbb{R}$
- $G : Z \rightrightarrows V$ multifunction, $G(z) := g(z) - C$.
- $g : Z \to V$ mapping, $C \subset V$ closed convex;

We consider the following tasks

- Given $\bar{z} \in g^{-1}(C)$, when is G metrically regular near $(\bar{z}, 0)$?
- If $\bar{z}$ denotes a solution of (P), what are first order optimality conditions?

Note

$$\bar{z} \in g^{-1}(C) \;\Leftrightarrow\; g(\bar{z}) \in C \;\Leftrightarrow\; 0 \in G(\bar{z})$$

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The linear case

Recall that the linear case also covers the differentiable case:

Theorem

Assume that g is continuously differentiable at $\bar{z}$ *and* $(\bar{z}, 0) \in \operatorname{gph} G$. *Then the following statements are equivalent:*

- *G is metrically regular near* $(\bar{z}, 0)$
- *The mapping* $g(\bar{z}) + Dg(\bar{z})(\cdot - \bar{z}) - C$ *is metrically regular near* $(\bar{z}, 0)$

In this section we assume

$$g(z) = Az - b,\ A \in L(Z, V),\ b \in V$$

G is certainly a convex multifunction:

$$\begin{aligned}
(z_i, v_i) \in \operatorname{gph} G, i = 1,2 \ &\Rightarrow \exists c_i \in C : v_i = Az_i - b - c_i,\ i = 1,2 \\
\Rightarrow \quad \alpha v_1 + (1-\alpha)v_2 &= \alpha(Az_1 - b - c_1) + (1-\alpha)(Az_2 - b - c_2) \\
&= A(\alpha z_1 + (1-\alpha)z_2) - b - \underbrace{(\alpha c_1 + (1-\alpha)c_2)}_{\in C},\ \forall \alpha \in [0,1] \\
\Rightarrow \quad \alpha(z_1, v_1) + (1-\alpha)(z_2, v_2) &\in \operatorname{gph} G,\ \forall \alpha \in [0,1]
\end{aligned}$$

Hence:

G metrically regular near $(\bar{z}, 0) \in \operatorname{gph} G$
$\Leftrightarrow \ 0 \in \operatorname{int} G(Z) = \operatorname{int}(AZ - b - C)$

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Linear case: equality constraints

- $C = \{0\}$
- $0 \in G(\bar{z}) \;\Leftrightarrow\; 0 \in A\bar{z} - b - \{0\} \;\Leftrightarrow\; A\bar{z} = b$
- G metr.reg.near $(\bar{z}, 0)$
  $\Leftrightarrow\; 0 \in \mathrm{int}\,(AZ - b) = \mathrm{int}\,(A(Z - \bar{z})) = \mathrm{int}\,(AZ)$
  $\Leftrightarrow A$ surjective
- Finite dimensional case: $Z = \mathbb{R}^n$, $V = \mathbb{R}^m$,
  A $m \times n$ matrix (representation of the linear operator)
  G metr.reg. $\Leftrightarrow A$ has full row rank m

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Infinite dimensional example

- $\Omega \subset \mathbb{R}^d$ bounded domain with $C^{0,1}$ boundary
- Find $z = (y, u)$ with $u \in L^2(\Omega)$ such that

$$\begin{aligned} -\triangle y - u &= 0, \text{ in } \Omega \\ y &= 0, \text{ on } \partial\Omega \end{aligned}$$

- Several possibilities for Z, V
- $Z = H_0^1(\Omega) \times L^2(\Omega)$, $V = H^{-1}(\Omega)$, $b = 0$
  A is given by

$$\langle Az, v \rangle = \langle A(y, u), v \rangle = \int_\Omega \nabla y \nabla v - \int_\Omega uv, \ \forall v \in H^1(\Omega)$$

  A is surjective since by the Lax-Milgram-Lemma for every $v \in H^{-1}(\Omega)$ there exists $y \in H_0^1(\Omega)$ such that

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Infinite dimensional example (cont.)

- $Z = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, $V = L^2(\Omega)$, A is given by

$$A(y,u) = -\sum_{i=1}^{d} \frac{\partial^2 y}{\partial x_i^2} - u$$

  Clearly, A is surjective.
  Under additional assumptions on Ω (e.g., $\partial\Omega \in C^{1,1}$), the system $Az = v$ has a solution of the form $z = (y,0)$ (will be important later)

- More generally, taking $Z = H_0^1(\triangle;\Omega) \times L^2(\Omega)$, $V = L^2(\Omega)$, where $H_0^1(\triangle;\Omega) := \{y \in H_0^1(\Omega) \,:\, -\triangle y \in L^2(\Omega)\}$,

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Generalization of Equality Constraints

- $C = \{0\}$ is a special case of $\mathrm{int}\, C = \emptyset$
- $0 \in A\bar{z} - b - C \Leftrightarrow A\bar{z} = b + c$ for some $c \in C$
- Assume A **surjective** $\Rightarrow$
  $0 \in \mathrm{int}\, A(Z - \bar{z}) = \mathrm{int}\,(AZ - b - c) \subset \mathrm{int}\,(AZ - b - C) \Rightarrow$
  G is metr.regular near $(\bar{z}, 0)$
- If $A = id$ then A is obviously surjective $\Rightarrow$ explicit constraints of the form $0 \in G(z) = z - C$ are always metrically regular

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where $\underline{\varphi}_u, \bar{\varphi}_u \in L^2(\Omega)$.

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- Then: A is surjective $\Rightarrow$ G is metrically regular.

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Outline

1. The problem
2. The linear case
   - Linear equality constraints
   - Linear Inequality constraints
   - Mixed Equality/Inequality constraints
3. The convex case
4. First-order optimality conditions

Inequality constraints

- We call $0 \in g(z) - C$ an *inequality constraint*, if $\operatorname{int} C \neq \emptyset$
- Typical finite dimensional example: $V = R^m$, $C = R^m_-$:

$$0 \in g(z) - \mathbb{R}^m_- \Leftrightarrow g(z) \leq 0$$

Theorem

Let $\operatorname{int} C \neq \emptyset$. $G(z) = Az - b - C$ is metrically regular near $(\bar{z}, 0) \in \operatorname{gph} G \Leftrightarrow$

$$\exists \tilde{z} : A\tilde{z} - b \in \operatorname{int} C \quad \text{(Slater condition)}$$

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Mixed Equality/Inequality systems

Theorem (Generalized Mangasarian-Fromovitz CQ)

Assume

- $V = V_1 \times V_2$
- $A = (A_1, A_2)$, $b = (b_1, b_2)$, $A_i : Z \to V_i$, $b_i \in V_i$, $i = 1, 2$
- $C = C_1 \times C_2$, $C_i \subset V_i$
- A_1 *is surjective,* $\operatorname{int} C_2 \neq \emptyset$

Then: G is metrically regular near $(\bar{z}, 0) \in \operatorname{gph} G \Leftrightarrow$

$$\exists \tilde{z} : A_1\tilde{z} - b_1 \in C_1 \;\wedge\; A_2\tilde{z} - b_2 \in \operatorname{int} C_2$$

Proof: Only "$\Leftarrow$": Let $B(A_2\tilde{z} - b_2, 2r) \subset C_2$, $B(0, \rho) \subset A_1 B(0, 1)$ and let $0 < R \leq r/\|A_2\|$. Then for any $(v_1, v_2) \in B(0, \rho R) \times B(0, r)$ there is some $h \in B(0, R) \subset Z$ such that $A_1 h = v_1$ and $\|A_2 h - v_2\| \leq 2r \Rightarrow$

$$v_1 \in A(\tilde{z}+h) - b_1 - C_1 \wedge A_2\tilde{z} - b_2 + (A_2 h - v_2) \in C_2 \Rightarrow (v_1, v_2) \in G(\tilde{z}+h)$$

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Example

- Find $z = (y, u)$ such that

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- Assume $\partial\Omega \in C^{1,1}$, $p \ge 2$, $d/2 < p < \infty$, $\underline{\varphi}_u, \bar{\varphi}_u, \underline{\varphi}_y, \bar{\varphi}_y \in L^p(\Omega)$
- Assume $\exists (\tilde{y}, \tilde{u}) \in H_0^1(\Omega) \times L^2(\Omega), \delta > 0$:

$$\begin{aligned} -\triangle \tilde{y} - \tilde{u} &= 0 \quad \text{in } \Omega, \\ \underline{\varphi}_u \le \tilde{u} &\le \bar{\varphi}_u \quad \text{a.e.in } \Omega, \\ \underline{\varphi}_y + \delta \le \tilde{y} &\le \bar{\varphi}_y - \delta \quad \text{a.e.in } \Omega \end{aligned}$$

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$$\begin{aligned} -\triangle \tilde{y} - \tilde{u} &= 0 \quad \text{in } \Omega, \\ \underline{\varphi}_u \le \tilde{u} &\le \bar{\varphi}_u \quad \text{a.e.in } \Omega, \\ \underline{\varphi}_y + \delta \le \tilde{y} &\le \bar{\varphi}_y - \delta \quad \text{a.e.in } \Omega \end{aligned}$$

Example

- Find $z = (y, u)$ such that

$$\begin{aligned} -\triangle y - u &= 0 \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \\ \underline{\varphi}_u \le u &\le \bar{\varphi}_u \quad \text{a.e.in } \Omega, \\ \underline{\varphi}_y \le y &\le \bar{\varphi}_y \quad \text{a.e.in } \Omega \end{aligned}$$

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- Set $Z = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \times L^p(\Omega)$,
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$$A_1(y, u) = \Big(-\sum_{i=1}^{d} \frac{\partial^2 y}{\partial x_i^2} - u, u\Big)$$

  $A_2(y, u) = y$
  Note that $A_1 : Z \to V_1$ is surjective since for every $f \in L^p(\Omega)$ the equation $-\triangle y = f$ has a unique solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$
- $C_1 = \{0\} \times \{\varphi \in L^p(\Omega) : \underline{\varphi}_u \leq \varphi \leq \bar{\varphi}_u\}$,
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The convex case

Here we assume that the mapping $G(z) = g(z) - C$ has a convex graph. Similar results as in the linear case are valid:

Theorem

Assume

- $V = V_1 \times V_2$
- $g = (g_1, g_2)$, $g_i : Z \to V_i$, $b_i \in V_i$, $i = 1, 2$
- $C = C_1 \times C_2$, $C_i \subset V_i$
- *G is a closed convex multifunction*
- $0 \in \operatorname{int}(g_1(Z) - C_1)$, $\operatorname{int} C_2 \neq \emptyset$
- $\exists \tilde{z} : \; g_1(\tilde{z}) \in C_1 \;\wedge\; g_2(\tilde{z}) \in \operatorname{int} C_2$

Then, G is metrically regular near $(\bar{z}, 0) \in \operatorname{gph} G$

Example for a convex multifunction

- Let Ω be a bounded domain.
- Let $A \in L(Z, L^2(\Omega)^d)$, $\bar{\varphi} \in L^2(\Omega)$, $\psi : \mathbb{R}^d \to \mathbb{R}$ convex, Lipschitz
- $g : Z \to L^2(\Omega)$, $g(z)(x) := \psi(Az(x))$
- $C = \{\varphi \in L^2(\Omega) \ : \ \varphi \leq \bar{\varphi} \text{a.e.}\}$
- Then G is convex.
- e.g. $Z = H_0^1(\Omega)$, $g(z) = |\nabla z|$.

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Optimality Conditions

For a convex set $C \subset V$ the *normal cone at* $\bar{c} \in C$ is given by

$$N_C(\bar{c}) = \{v^* \in V^* \;:\; \langle v^*, c - \bar{c}\rangle \leq 0, \forall c \in C\}$$

Theorem

Let $\bar{z}$ be a solution for the problem (P) and assume that g and J are continuously differentiable at $\bar{z}$ and that G is metrically regular near $(\bar{z}, 0)$. Then there is a multiplier $v^* \in N_C(g(\bar{z}))$ such that

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Normal cones

- $N_{C_1 \times C_2}(\bar{c}_1, \bar{c}_2) = N_{C_1}(\bar{c}_1) \times N_{C_2}(\bar{c}_2)$
- $N_{\{c\}}(c) = V^*$
- $N_C(\bar{c}) = \{0\}$ if $\bar{c} \in \operatorname{int} C$
- $C = (-\infty, a] \Rightarrow$

$$N_C(\bar{c}) = \begin{cases} \mathbb{R}^+ & \text{if } \bar{c} = a \\ 0 & \text{if } \bar{c} < a \end{cases}$$

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$$\min_{z \in H_0^1(\Omega)} J(z) := \frac{1}{2}\int_\Omega |\nabla y|^2 + \langle f, z \rangle$$

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- $Z = Y \times L^p(\Omega)$, $V = L^p(\Omega) \times L^p(\Omega) \times Y$, $Y := (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$
- Optimality conditions: There exist multipliers $(p^*, \mu^*, \nu^*) \in L^q(\Omega) \times L^q(\Omega) \times Y^*$ such that

$$\int_\Omega (\bar{y} - y_d)y + \langle -\triangle^* p^*, y\rangle + \langle \nu^*, y\rangle = 0, \forall y \in Y,\ \bar{u} - p^* + \mu^* = 0,$$

conditions on μ^* as before and

$$\langle \nu^*, y - \bar{y}\rangle \le 0, \quad \forall y \in Y : \underline{\varphi}_y \le y \le \bar{\varphi}y$$

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$$\int_\Omega (\bar{y}-y_d)y + \langle -\triangle^* p^*, y\rangle + \langle \nu^*, y\rangle = 0, \forall y \in Y,\ \bar{u} - p^* + \mu^* = 0,$$

conditions on μ^* as before and

$$\langle \nu^*, y - \bar{y}\rangle \leq 0, \quad \forall y \in Y : \underline{\varphi}_y \leq y \leq \bar{\varphi}_y$$

Control problem 2

$$\begin{aligned}
\min_{z=(y,u)\in Z} J(y,u) &:= \frac{1}{2}\int_\Omega ((y-y_d)^2+u^2)\\
s.t. \quad -\triangle y - u &= 0\\
\underline{\varphi}_u \leq u &\leq \bar{\varphi}_u\\
\underline{\varphi}_y \leq y &\leq \bar{\varphi}_y
\end{aligned}$$

- $Z = Y \times L^p(\Omega)$, $V = L^p(\Omega)\times L^p(\Omega)\times Y$, $Y := (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))$
- Optimality conditions: There exist multipliers $(p^*,\mu^*,\nu^*)\in L^q(\Omega)\times L^q(\Omega)\times Y^*$ such that

  $$\int_\Omega(\bar{y}-y_d)y + \langle -\triangle^* p^*, y\rangle + \langle \nu^*, y\rangle = 0, \forall y\in Y,\ \bar{u}-p^*+\mu^* = 0,$$

  conditions on μ^* as before and

  $$\langle \nu^*, y-\bar{y}\rangle \leq 0, \quad \forall y \in Y : \underline{\varphi}_y \leq y \leq \bar{\varphi}_y$$