Seminar on Numerical Analysis

Chhitiz Buchasia

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A General Superlinear Convergence Result

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2. We will look at the minimum requirements for the operator G.

Application to Obstacle Problem

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$$\Updownarrow$$

Constrained Minimization Problem: Find

$$u \in U : \mathfrak{J}(u) = \inf_{v \in U} \mathfrak{J}(v), \tag{2}$$

where, $U := \{v \in H_0^1(\Omega) : v(x) \geq g(x) \dot{\forall} x \in \Omega\}$with,

$\mathfrak{J}(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega fv dx$ and given $f \in L^2(\Omega)$,

$g \in \{v \in H^1(\Omega) : v(x) \leq 0, \dot{\forall} x \in \Gamma\}$

The minimization problem is equivalent to the Variational Inequality(VI): Find $u \in U$:

$$\int_\Omega \nabla^T u \nabla (v-u) dx \geq \int_\Omega f(v-u)dx, \quad \forall v \in U \tag{3}$$

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This Variational Inequality is equivalent to the nonlinear fixed point equation. Find $u \in X$:

$$u = B_g(u) := P[(I - \rho \mathcal{J}^{-1} A)u + \rho \mathcal{J}^{-1} f] \tag{4}$$

in X

Application to Optimal Control

Consider the following optimal control problem:

$$min_{y \in H_0^1(\Omega), u \in L^2(\Omega)} \mathfrak{J}(y, u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \quad (5)$$

such that $Ay = u, \quad \beta_l \leq u \leq \beta_r$

Here, $y \in H_0^1(\Omega)$ is the state, which is defined in Ω
$\Omega \subset \mathbb{R}^n$ is open bounded domain,
$u \in L^2(\Omega)$ is the control,
$A : H_0^1(\Omega) \to H^{-1}(\Omega) = H_0^1(\Omega)^*$ is a linear elliptic partial differential operator

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This can be rewritten as

$$\Phi(u) = u - P_{[\beta_l, \beta_r]}(u - \theta \nabla \hat{\mathfrak{J}}(u)) = 0. \tag{7}$$

Algorithm: Generalized Newton's Method

Consider the operator equation (1) with $G : X \to Y$, X, Y Banach spaces. Algorithm for Generalized Newton's Method:

1. Choose $x^0 \in X$
   For $k = 0, 1, 2, \ldots$
2. Choose an invertible operator $M_k \in L(X, Y)$.
3. Obtain s^k by solving
   $$M_k s^k = -G(x^k)$$
   and set $x^{k+1} = x^k + s^k$.

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1. (x^k) converges q-linearly to $\bar{x}$ with rate $\gamma \in (0,1)$ iff

$$\|M_k^{-1}(G(\bar{x}+d^k) - G(\bar{x}) - M_k d^k)\|_X \leq \gamma \|d^k\|_X \qquad (8)$$

   $\forall k$ with $\|d^k\|_X$ sufficiently small.

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2. (x^k) converges q-superlinearly to $\bar{x}$ iff

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   for $\|d^k\|_X \to 0$

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3. (x^k) converges q-order $1+\alpha > 1$ to $\bar{x}$ iff

$$\|M_k^{-1}(G(\bar{x}+d^k) - G(\bar{x}) - M_k d^k)\|_X = O(\|d^k\|_X^{1+\alpha}) \tag{10}$$

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1. Regularity condition:

$$\|M_k^{-1}\|_{Y\to X} \le C, \quad \forall k \ge 0. \tag{11}$$

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1. Regularity condition:

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2. Approximation condition:

   $\|G(\bar{x}+d^k)-G(\bar{x})-M_k d^k\|_Y = o(\|d^k\|_X)$

   or $\|G(\bar{x}+d^k)-G(\bar{x})-M_k d^k\|_Y = O(\|d^k\|_X^{1+\alpha})$

Theorem 1. Consider the operator equation (1) with $G : X \to Y$, where X and Y are Banach spaces. Let (x^k) be generated by the generalized Newton method. Then:

1. If x^0 is sufficiently close to $\bar{x}$ and (8) holds then $x^k \to \bar{x}$ q-linearly with rate γ.
2. If x^0 is sufficiently close to $\bar{x}$ and (9) holds then $x^k \to \bar{x}$ q-superlinearly.
3. If x^0 is sufficiently close to $\bar{x}$ and (10) holds then $x^k \to \bar{x}$ q-superlinearly with order $1 + \alpha$.

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2. The fast convergence of the generalized Newton's method is not affected if the system is solved inexactly and the accuracy of the solution is controlled suitably.

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3. The regularity condition then reads

$$\|G'(x^k)^{-1}\|_{Y \to X} \leq C, \quad \forall k \geq 0$$

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2. If G' is α-order Hölder continuous near $\bar{x}$, we can prove approximation condition of order $1+\alpha$.

Finally we have the following

Corollary 2 Let $G : X \to Y$ be a continuously F-differential operator between Banach spaces and assume that $G'(\bar{x})$ is continuously invertible at the solution $\bar{x}$. Then Newton's method (i.e. Algorithm for Generalized Newton's Method with $M_k = G'(x^k)$ for all k) converges locally q-superlinearly. If, in addition, G' is α-order Hölder continuous near $\bar{x}$, the order of convergence is $1 + \alpha$.

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3. Algorithm for Semismooth Newton Method.
4. Convergence Results.

THANK YOU!!!