Lemma

Let V be a Hilbert space with inner product $(\cdot,\, \cdot)$ and the associated norm $\|\cdot\|$. Let $M : V \to V$ a linear, bounded, and self-adjoint operator. Then

$$\|M\| := \sup_{v\in V\setminus\{0\}} \frac{\|Mv\|}{\|v\|} = \sup_{v\in V\setminus\{0\}} \frac{|(Mv,\, v)|}{(v,\, v)}.$$

Proof.
We prove the equality by showing both "$\geq$" and "$\leq$".
"$\geq$": Using Cauchy's inequality one easily shows

$$\|Mv\| = \sup_{y\in V\setminus\{0\}} \frac{|(Mv,\, y)|}{\|y\|}\,.$$

Hence,

$$\|M\| = \sup_{v\in V\setminus\{0\}} \sup_{y\in V\setminus\{0\}} \frac{(Mv,\, y)}{\|v\|\,\|y\|} \geq \sup_{v\in V\setminus\{0\}} \frac{|(Mv,\, v)|}{\|v\|^2}$$

"$\leq$": For an arbitrary real number $\beta > 0$ we have

$$\begin{aligned}
&4\,\|Mv\|^2 \\
&= \big(M(\beta\, v + \beta^{-1}Mv),\ \beta\, v + \beta^{-1}Mv\big) - \big(M(\beta\, v - \beta^{-1}\, Mv),\ \beta\, v - \beta^{-1}\, Mv\big) \\
&\leq \sup_{x\in V\setminus\{0\}} \frac{|(Mx,\, x)|}{(x,\, x)} \Big\{\|\beta\, v + \beta^{-1}Mv\|^2 + \|\beta\, v - \beta^{-1}Mv\|^2\Big\} \\
&= 2 \sup_{x\in V\setminus\{0\}} \frac{|(Mx,\, x)|}{(x,\, x)} \Big\{\beta^2\,\|v\|^2 + \beta^{-2}\,\|Mv\|^2\Big\}.
\end{aligned}$$

The last inequality follows from the parallelogram law.
Choosing $\beta = \sqrt{\|Mv\|/\|v\|}$ we obtain

$$4\,\|Mv\|^2 \leq 2 \sup_{x\in V\setminus\{0\}} \frac{|(Mx,\, x)|}{(x,\, x)} \Big\{2\|Mv\|\,\|v\|\Big\}$$

and so

$$\|Mv\| \leq \sup_{x\in V\setminus\{0\}} \frac{|(Mx,\, x)|}{(x,\, x)}\,\|v\|\,,$$

which implies "$\leq$". □

Source: H. Heuser: *Funktionalanalysis*, B. G. Teubner, Stuttgart 1992.