Why is Jacobi so bad?

Consider our 1D model problem

$$-u''(x) = f(x) \quad \forall x \in (0,\, 1), \qquad u(0) = g_0, \qquad u'(1) = g_1$$

Obviously, the entire solution depends on g_0 ("feature" of elliptic PDEs).
In particular, $u_h(1) = u_{n_h}$ depends on g_0.

"Homogenization:" $u_h^{(k)} \;\in\; g_0\,\varphi_0 + V_0$

Set $\underline{u}_h^{(0)} = \vec{o}$, i. e., $u_h^{(0)} = g_0\,\varphi_0$

Jacobi-preconditioned Richardson method:

- $$\underline{r}_h^{(0)} \;=\; \underline{f}_h - K_h\,\vec{o} \;=\; \begin{pmatrix} f_1^{(1)} + f_0^{(2)} \\ f_1^{(2)} + f_0^{(3)} \\ \vdots \\ f_1^{(n_h-1)} + f_0^{(n_h)} \\ f_1^{(n_h)} \end{pmatrix} + \begin{pmatrix} g_0/h_1 \\ 0 \\ \vdots \\ 0 \\ g_1 \end{pmatrix}$$
- $\underline{u}_h^{(1)} = \underline{u}_h^{(0)} + \tau D_h^{-1} \underline{r}_h^{(0)}$

  D_h diagonal $\quad\rightsquigarrow\quad$ g_0 appears in $u_1^{(1)}$ only!
- $\underline{r}_h^{(1)} = \underline{f}_h - K_h\,\underline{u}_h^{(1)}$

  $$K_h = \begin{pmatrix} * & * & 0 & \dots & 0 \\ * & * & * & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & * & * & * \\ 0 & \dots & 0 & * & * \end{pmatrix} \quad\rightsquigarrow\quad g_0 \text{ appears in } r_1^{(1)} \text{ and } r_2^{(1)} \text{ only!}$$
- $\underline{u}_h^{(2)} = \underline{u}_h^{(1)} + \tau D_h^{-1} \underline{r}_h^{(1)} \quad\rightsquigarrow\quad g_0$ appears in $u_1^{(2)}$ and $u_2^{(2)}$ only!

  $\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$

  $\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$
- $\underline{u}_h^{(k)} = \underline{u}_h^{(k-1)} + \tau D_h^{-1} \underline{r}_h^{(k-1)} \quad\rightsquigarrow\quad g_0$ appears in $u_1^{(k)}, \dots, u_k^{(k)}$ only!

Summary:
We need $k = n_h$ steps until the influence of g_0 is present in $u_h^{(k)}(1)$.

The information transfer is rather slow!

Idea:
Choose preconditioner C_h such that the information can propagate faster.
$\rightsquigarrow$ *multi-level* and *multi-grid* methods.