The "missing" interpolation error estimate

Let $I_h : V \longrightarrow V_h$ denote the interpolation operator. We want to estimate $\|v - I_h v\|_{L^2(T_k)}$ in terms of v'. Transformation to the reference element yields

$$\int_{T_k} |v(x) - (I_h v)(x)|^2 \, dx \;=\; h_k \int_{\widehat{T}} |\hat{v}(\xi) - \hat{I}\hat{v}(\xi)|^2 \, d\xi \,,$$

where $\hat{v} = v \circ F_k$ and $\hat{I}$ is the interpolation operator on the reference element. From what we have shown in the lecture we can conclude that

$$\begin{aligned} \hat{v}(\xi) - \hat{I}\hat{v}(\xi) \;&=\; \int_0^\xi \hat{v}'(z)\,dz - \xi \int_0^1 \hat{v}'(z)\,dz \\ &=\; (1-\xi)\int_0^\xi \hat{v}'(z)\,dz - \xi \int_\xi^1 \hat{v}'(z)\,dz \end{aligned}$$

Using the elementary formula $(a+b)^2 \le 2\,a^2 + 2\,b^2$ and Cauchy's inequality, we obtain

$$\begin{aligned} \int_{\widehat{T}} |\hat{v}(\xi) - \hat{I}\hat{v}(\xi)|^2 \, d\xi \;&\le\; \int_0^1 \Big\{ 2\,(1-\xi)^2 \Big(\int_0^\xi \hat{v}'(z)\,dz \Big)^2 + 2\,\xi \Big(\int_\xi^1 \hat{v}'(z)\,dz \Big)^2 \Big\}\, d\xi \\ &\le\; \int_0^1 2\,(1-\xi)^2\,\xi\,\|\hat{v}'\|^2_{L^2(0,\xi)} + 2\,\xi\,(1-\xi)\,\|\hat{v}'\|^2_{L^2(\xi,1)}\, d\xi \\ &\le\; \underbrace{\int_0^1 2\,(1-\xi)\,\xi\,d\xi}_{=1/3}\; \|\hat{v}'\|^2_{L^2(0,1)} \,, \end{aligned}$$

where in the last step we used that $\max\{1-\xi,\,1\} = 1$. Transforming back to the real element yields

$$\int_{\widehat{T}} |\hat{v}(\xi) - \hat{I}\hat{v}(\xi)|^2 \, d\xi \;\le\; \frac{1}{3} \int_{\widehat{T}} |\hat{v}'(\xi)|^2 \, d\xi \;=\; \frac{1}{3}\,\frac{h_k^2}{h_k} \int_{T_k} |v'(x)|^2 \, dx \,,$$

where $1/h_k$ comes from the transformation itself, h_k^2 comes from $v'(x) = \frac{1}{h_k}\,\hat{v}'(\xi)$ for $x = F_k(\xi)$.

Putting everything together we arrive at the estimate

$$\|v - I_h v\|^2_{L^2(T_k)} \;\le\; \frac{1}{3}\,\underbrace{h_k^2}_{\le h^2}\,\|v'\|^2_{L^2(T_k)} \qquad \forall v \in H^1(T_k)\,.$$

which holds because of a closure argument (C^1 is dense in H^1 w.r.t. the H^1-norm). Summing over the elements yields the global estimate

$$\|v - I_h v\|_{L^2(0,\,1)} \;\le\; \frac{h}{\sqrt{3}}\,\|v'\|_{L^2(0,\,1)} \qquad \forall v \in H^1(0,\,1)\,.$$

Compared to the estimate from the lecture,

$$\|v - I_h v\|_{L^2(0,\,1)} \;\le h^2\,\|v''\|_{L^2(0,\,1)} \qquad \forall v \in H^2(0,\,1)\,,$$

we have one power of h less, but also less smoothness assumptions on u.