Improved Euler / Explicit midpoint rule

$$\begin{aligned} g_1 &= u_j \\ g_2 &= u_j + \tfrac{\tau_j}{2}\, f(t_j,\, g_1) \\ u_{j+1} &= u_j + \tau_j\, f(t_j + \tfrac{\tau_j}{2},\, g_2) \end{aligned}$$

In short:

$$u_{j+1} \;=\; u_j + \tau_j\,\phi(t_j,\, u_j,\, \tau_j) \qquad \text{where} \qquad \phi(t,\, u,\, \tau) := f\big(t + \tfrac{\tau}{2},\, u + \tfrac{\tau}{2} f(t,u)\big)$$

$$\tfrac{1}{\tau_j}\big(u_{j+1} - u_j\big) \;=\; \phi(t_j,\, u_j,\, \tau_j)$$

Consistency analysis

Local error: $d_\tau(t+\tau) := u(t+\tau) - \big[u(t) + \tau\,\phi(t,\, u(t),\, \tau)\big]$
Consistency error: $\psi_\tau(t_k) = \psi_k$ with

$$\begin{aligned} \psi_{j+1} &= \tfrac{1}{\tau_j}\big(u(t_{j+1}) - u(t_j)\big) - \phi(t_j,\, u(t_j),\, \tau_j) \\ \psi_0 &= u(0) - u_0 \;=\; 0 \end{aligned}$$

For simplicity: $n = 1$, assume that f is sufficiently smooth

$$\begin{aligned} d_\tau(t+\tau) \;&=\; u(t+\tau) - \big[u(t) + \tau\,\phi(t,\, u(t),\, \tau)\big] \\ &= u(t+\tau) - u(t) - \tau\, f\big(t + \tfrac{\tau}{2},\, u(t) + \tfrac{\tau}{2} f(t,\, u(t))\big) \\ &= u(t) + \tau\, u'(t) + \tfrac{\tau^2}{2} u''(t) + \tfrac{\tau^3}{6} u'''(t) + \mathcal{O}(\tau^4) - u(t) \\ &\quad - \tau\left[\big(f + \tfrac{\tau}{2} f_t + \tfrac{\tau}{2} f_u\, f + \tfrac{\tau^2}{4} f_{tt} + \tfrac{\tau^2}{2} f_{tu}\, f + \tfrac{\tau^2}{4} f_{uu}\, f^2\big)_{(t,\,u(t))} + \mathcal{O}(\tau^3)\right] \\ &= \tau\Big[\underbrace{u'(t) - f(t,\, u(t))}_{=0}\Big] + \tfrac{\tau^2}{2}\Big[\underbrace{u''(t) - \big(f_t - f_u\, f\big)_{(t,\,u(t))}}_{=(*)=0}\Big] + \\ &\quad + \tau^3\Big[\tfrac{1}{6} u'''(t) - \tfrac{1}{4}\big(f_{tt} + 2 f_{tu}\, f + f_{uu}\, f^2\big)_{(t,\,u(t))}\Big] + \mathcal{O}(\tau^4) \end{aligned}$$

where

$$0 \;=\; \frac{d}{dt}\Big[u'(t) - f(t,\, u(t))\Big] \;=\; u''(t) - \big(f_t(t,\, u(t)) + f_u(t,\, u(t))\, \underbrace{u'(t)}_{=f(t,u(t))}\,\big) \;=\; (*)$$

Recall that $\psi_\tau(u)(t_{j+1}) = \frac{1}{\tau_j}\, d_\tau(t_{j+1})$. A detailled analysis shows that

$$\exists K > 0 :\; \|\psi_\tau(t)\|_{Y_\tau} \;\leq\; K\,\tau^2$$

$\Longrightarrow$ The improved Euler method has **consistency order** 2.