Lemma 1.56

If $r_{k-1} \neq 0$:

(a) $p_{k-1} \neq 0$

(b) $\mathcal{K}_k(A, r_0) \;=\; \mathrm{span}(r_0, \ldots, r_{k-1}) \;=\; \mathrm{span}(p_0, \ldots, p_{k-1})$

(c) $(r_k, p_j) \;=\; 0 \qquad \forall j = 0, 1, \ldots, k-1$

(d) $(Ap_k, p_j) \;=\; 0 \qquad \forall j = 0, 1, \ldots, k-1$

Proof by induction.

For $k = 1$ the statements are trivial or follow from (18), (19).

Suppose that (a)–(d) hold for k and assume that $r_k \neq 0$.

$$r_k \;=\; r_{k-1} - \alpha_k\, Ap_{k-1} \;\in\; \mathcal{K}_k(A, r_0) + A\big(\mathcal{K}_k(A, r_0)\big) \;\subset\; \mathcal{K}_{k+1}(A, r_0).$$

Proof of (a) and (b).

From (b) and (c) we know that $r_k \perp \mathcal{K}_k(A, r_0)$, which implies that

$$\mathcal{K}_k(A, r_0) \;\subsetneq\; \mathrm{span}(r_0, \ldots, r_k) \;\subset\; \mathcal{K}_{k+1}(A, r_0).$$

However, $\dim(\mathcal{K}_{k+1}(A, r_0)) = \dim(\mathcal{K}_k(A, r_0)) + 1$. Therefore,

$$\mathrm{span}(r_0, \ldots, r_k) \;=\; \mathcal{K}_{k+1}(A, r_0).$$

From the definition of the algorithm we know that $r_k = p_k - \beta_{k-1}\, p_{k-1}$. Hence,

$$\mathrm{span}(p_0, \ldots, p_k) \;=\; \mathrm{span}(p_0, \ldots, p_{k-1}, r_k) \;\overset{(b)}{=}\; \mathrm{span}(r_0, \ldots, r_k).$$

This means, (a) and (b) hold for $k+1$.

Proof of (c).

From formula (18) we know that $(r_{k+1}, p_j) = 0$ for $j = k$. For $j < k$,

$$(r_{k+1}, p_j) \;=\; (r_k - \alpha_k\, Ap_k, p_j) \;=\; (r_k, p_j) - \alpha_k (Ap_k, p_j) \;\overset{(c),(d)}{=}\; 0.$$

Thus, $r_{k+1} \perp \mathrm{span}(p_0, \ldots, p_k) = \mathcal{K}_{k+1}(A, r_0)$ and so (c) holds for $k+1$.

Proof of (d).

From formula (19) we know that $(Ap_{k+1}, p_j) = 0$ for $j = k$. For $j < k$,

$$(Ap_{k+1}, p_j) \;=\; (p_{k+1}, Ap_j) \;=\; \underbrace{(r_{k+1}, \underbrace{Ap_j}_{\in \mathcal{K}_{k+1}(A, r_0)})}_{=0} + \beta_k\, \underbrace{(p_k, Ap_j)}_{\overset{(d)}{=}0} \;=\; 0.$$

This means (d) holds for $k+1$, which completes the proof. $\square$